COLLEGE ALGEBRA

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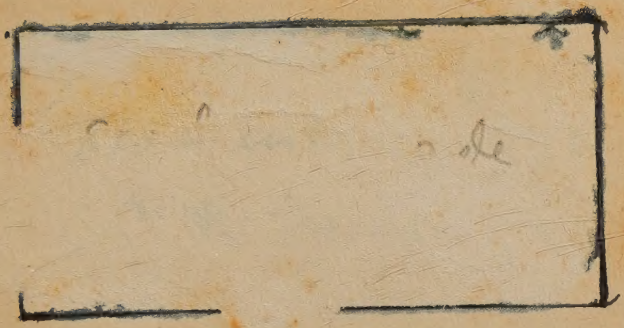

D.B. Marion
E.H.C.

x = outer edge

$x+2$ = inner edge.

$(x+2)(x-2)+(x-2)$

x^2+4x+4

$x+2$

x^3+4x^2+4x

$2x^2+8x+4$

$x^3+6x^2+12x+4$

COLLEGE ALGEBRA

BY

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PREFACE

An effort has been made to present here the elementary principles of algebra in a simple and direct way, and to give rigorous proofs of the theorems used. It seemed desirable, however, to pass over certain delicate points that are beyond the comprehension of the student, and to omit altogether several difficult proofs. The treatment is concrete, and graphical methods have been used freely.

The introduction to complex numbers that is commonly given seems to me to be arbitrary and unconvincing; and I have sought by putting forward the concrete, geometrical side of the question and by careful attention to the proper sequence of the ideas introduced to present these numbers in a rational and convincing way.

Students almost invariably think that the symbol ∞ represents a number called "infinity," just as the symbol 5 represents a number called "five," and this view is fostered, to some extent at least, by the conventional use of the word "infinity," and by the employment of a symbol to represent it. It seemed worth while, therefore, to depart from this conventional treatment and to write the chapter on infinite series without using either the symbol for infinity or the word itself, except as it is implied in the phrase "infinite series."

I have selected the problems with a view to convincing the student that algebra is a body of principles by the aid of which certain kinds of important information can be obtained from data that do not give this information explicitly. Many problems have been selected that illustrate the simpler principles of physics, geometry, and analytic geometry. The analytic geometry problems are somewhat of an innovation in a text of

this kind, and objection may be made that the difficulties of this subject ought not to be added to those inherent in the algebra. But there is a decided advantage in making the student feel that mathematics is one.

For the sake of adding to the flexibility of the book two proofs of the Binomial Theorem have been given, one in the chapter on Permutations and Combinations, and the other in the chapter on Mathematical Induction. Either chapter may be omitted without the necessity of omitting the Binomial Theorem.

Attention is called to the examples given in the text of the chapter on Mathematical Induction to show the necessity of both parts of the proof by induction.

Articles 134–138 should be omitted by students who have not studied trigonometry.

I am greatly indebted to Professors D. R. Curtiss of Northwestern University, W. B. Carver of Cornell University, and R. G. D. Richardson and H. P. Manning of Brown University for many suggestions of importance and value.

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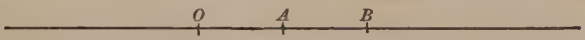
CHAPTER I

THE FUNDAMENTAL OPERATIONS

1. The operations of addition, subtraction, multiplication, and division are called the fundamental operations of algebra. We shall assume that the student is familiar with the details of performing these operations and shall confine our attention here to a brief consideration of their important properties.

2. The representation of points by numbers. — Since the letters used in algebra represent numbers, it is of great importance for the student to have a clear conception of the nature of numbers. He will acquire this best by thinking of numbers as representatives of points.

Consider an indefinite straight line called the **axis**, a fixed point or **origin** O on this line, and a given length ———, which we shall call the unit length.



We shall let the point A which is on the line at a distance to the right of the origin equal to this unit length be represented by the number 1. The point B twice this distance to the right of the origin will be represented by the number 2, and similarly any point to the right of the origin on this line is represented by the number which gives its distance from the origin in terms of the given unit of length. The points to the left of the origin on the line are also represented by the numbers which give their respective distances from the origin. But it is necessary to distinguish these numbers in some way

from the representatives of the points to the right of the origin. This is done by prefixing the minus sign to these numbers, and then to make this general scheme more symmetrical, we should prefix a plus sign to the representatives of the points to the right of the origin. Thus, -5 represents the point five units to the left of the origin, while $+5$ represents the point five units to the right of the origin. The two points are the same distance from the origin, but in opposite direction from it. It is this difference that is indicated by the two opposite signs.

Definitions. — The numbers that represent the points on the left of the origin are called **negative** numbers and those that represent the points on the right of the origin are called **positive** numbers.

The plus sign that is a part of the positive numbers is usually not written since its omission causes no ambiguity.

These positive and negative numbers, together with the number 0, which represents the origin, are called **real** numbers for a reason that is explained in Chapter XII.

The quotient of two integers is called a **rational** number. Every integer is a rational number since it is the quotient of itself and 1.

The number that represents a point is called the **abscissa** of this point.

The distance of a point from the origin is called the **absolute value** of the number that represents the point.

Thus, two numbers that represent points on opposite sides of the origin but equidistant from it have the same absolute value; as, for example, the numbers $+7$ and -7. The absolute value of these numbers is 7.

3. Meaning of "greater than" and "less than." — If the point A has the abscissa a and is to the right of the point B which has the abscissa b, a is said to be greater than b, and b less than a. Thus every positive number is greater than 0 and every negative number is less than 0. Moreover every positive number is greater than any negative number. For ex-

ample, 5 is greater than -6. This relationship is indicated thus; $5 > -6$, or $-6 < 5$. These statements are read "5 is greater than -6" and "-6 is less than 5" respectively.

We express the fact that the numbers a and b represent different points by the symbol $a \neq b$, which is read, "a is not equal to b."

4. Definition of sum. — If A and B are any points on the axis and if the segment AC equals the segment OB in magnitude and direction, the abscissa c of the point C is called the **sum** of the abscissæ a and b of the points A and B respectively.

This connection of the numbers a, b, and c is expressed thus:

$$a + b = c.$$

The process of finding the sum of two numbers is called **addition.**

By the sum of three numbers a, b, and c we mean the sum of $a + b$ and c.

If we represent this sum by $a + b + c$, then $a + b + c = (a + b) + c$.

5. Assumptions concerning real numbers and the operation of addition. — We shall make the following assumptions concerning real numbers and the operation of addition. A careful inspection of a figure will convince the student of the appropriateness of most of these assumptions.

I. *If a and b are any numbers, there is one and only one number x such that $a + b = x$.*

II. *Addition is commutative.*

This means that $a + b = b + a$.

III. *The sum of three numbers is the same, irrespective of the way in which they are grouped.*

Thus, $a + b + c = (a + b) + c = a + (b + c)$.

This is the **associative law of addition.**

IV. *If* $a + b = a + c$, *then* $b = c$.

This is the **law of cancellation for addition.**

V. *There is one and only one number* x *such that* $x + x = x$.

This number is 0.

VI. *For every number* a, *there is one and only one number* $(-a)$ *such that* $a + (-a) = 0$.

If a is a negative number, it is clear that the $(-a)$ of this assumption must be a positive number.

The familiar statement that if equal numbers be added to equal numbers the sums are equal numbers is equivalent to Assumption I.

6. Consequences of the assumptions. — The following theorems are consequences of these assumptions.

1. *If* a *is any number,* $a + 0 = a$.

2. *If* a *and* b *are any numbers, there is one and only one number* x *such that* $a = b + x$.

Definition. — The process of finding x when a and b are given is called the process of **subtracting** b from a. In this process a is called the **minuend**, b the **subtrahend**, and x the **difference** or **remainder.**

Subtraction is indicated by the minus sign.

Thus, the relation of a, b, and x just described is expressed by the equation

$$a - b = x.$$

The process of subtracting b from a is equivalent to that of finding the length and direction of the segment BA, A and B being the points whose abscissæ are a and b respectively.

The statement that if equal numbers be subtracted from equal numbers the remainders are equal numbers is equivalent to Theorem 2.

3. *The result of adding* $(-b)$ *to any number* a *is the same as the result of subtracting* b *from* a.

That is, $$a + (-b) = a - b.$$

4. *The result of subtracting* $(-b)$ *from any number* a *is equal to the result of adding* b *to* a.

That is, $$a - (-b) = a + b.$$

Although Theorems 1–4 are consequences of Assumptions I–V, they seem so obvious from the definition of addition and an inspection of a figure that we shall omit the formal proofs.

The familiar rule for finding the difference of two numbers, — namely, to change the sign of the subtrahend and add the resulting number to the minuend, — is an immediate consequence of Theorems 3 and 4.

7. Parentheses. — An expression like $a-b+c+d$ indicates that $(-b)$ is to be added to a, c added to the resulting sum, and d to this last sum. That is, $a-b+c+d$ is a simplified form for $[(a-b)+c]+d$. Now

$$\begin{aligned}[(a-b)+c]+d &= (a-b)+(c+d) \qquad \text{(III)}\\ &= a+[-b+(c+d)]\\ &= a+[-b+c+d].\end{aligned}$$

Hence,
$$\begin{aligned}x+(a-b+c+d) &= x+(a+[-b+c+d])\\ &= (x+a)+(-b+c+d)\\ &= (x+a-b)+(c+d)\\ &= (x+a-b+c)+d\\ &= x+a-b+c+d.\end{aligned}$$

This procedure is obviously applicable to parentheses containing any number of terms. Hence,

5. *An expression involving parentheses preceded by the plus sign is unchanged if the parentheses are removed and the inclosed terms are left with their original signs.*

Consider the expression

$$a-(b+c-d).$$

If $a-(b+c-d)=x,$

then $a=x+(b+c-d),$ (Definition of Subtraction.)

and $a=x+b+c-d.$ (5)

If we subtract b and c from each member of this equation and add d to each member, we get

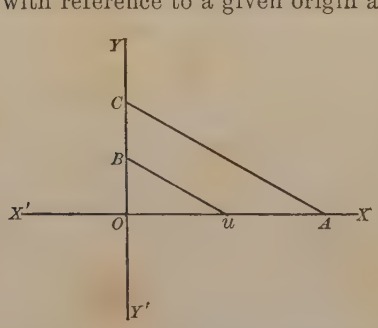
$$a-b-c+d=x.$$

Hence, $a-(b+c-d)=a-b-c+d.$

This procedure is obviously applicable to any parentheses preceded by the minus sign. Hence,

6. *An expression involving parentheses preceded by the minus sign is unchanged if the parentheses are removed and the sign of each of the inclosed terms is changed.*

8. Definition of multiplication. — We have seen that the real numbers are the abscissæ of the points on an indefinite line with reference to a given origin and a given unit distance.



Let this line be $X'X$, the origin O, and the unit distance OU.

Let $Y'Y$ be an indefinite line through O perpendicular to $X'X$. The real numbers will also be the abscissæ of the points of this line with reference to the same origin and the same unit distance, the positive numbers representing the points above the origin.

Let a and b be any two numbers, and let A be the point of $X'X$ whose abscissa is a, and B the point of $Y'Y$ whose abscissa is b. Draw UB and then AC parallel to UB. The number c which is the abscissa of the point C on $Y'Y$ is called the *product* of a and b.

This connection of a, b, and c is expressed thus:

$$a\cdot b=c;\ \text{or thus, } a\times b=c;\ \text{or thus, } ab=c.$$

The process of finding the product of two numbers is called **multiplication**, and the numbers are called **factors**.

By the product of three numbers a, b, and c we mean the product of ab and c.

If we represent this product by abc, then $abc = (ab)c$.

9. Assumptions concerning real numbers and the operation of multiplication. — We shall make the following assumptions concerning real numbers and the operation of multiplication. A careful inspection of a figure will convince the student of the appropriateness of most of these assumptions.

VII. *If a and b are any numbers, there is one and only one number x such that $ab = x$.*

VIII. *Multiplication is commutative.*

This means that $ab = ba$.

IX. *The product of three numbers is the same irrespective of the way in which they are grouped.*

Thus, $(ab)c = a(bc)$.

This is the **associative law of multiplication.**

X. *If $ab = ac$, and $a \neq 0$, then $b = c$.*

This is the **law of cancellation** for multiplication.

XI. *The number 1 is such that $1 \cdot 1 = 1$.*

XII. *For every number a not equal to 0 there is one and only one number a' such that $a \cdot a' = 1$.*

The number a' is called the **reciprocal** of a, and a, the reciprocal of a'.

XIII. *Multiplication is distributive as to addition.*

This means that $a(b + c) = ab + ac$.

XIV. *The product of two positive numbers is a positive number.*

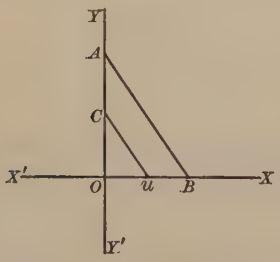
The familiar statement that if equal numbers be multiplied by equal numbers the products are equal numbers is equivalent to Assumption VII.

10. Consequences of the assumptions. — The following theorems are consequences of these assumptions:

7. *If a is any number,* $a \cdot 1 = a$.

This is obvious from the figure.

8. *If a and b are any numbers and b is not equal to 0, there is one and only one number x such that $a = bx$.*



Let U and B be the points of $X'X$ whose abscissæ are 1 and b respectively, and let A be the point of $Y'Y$ whose abscissa is a. Then if we draw the line through U parallel to BA, it will cut $Y'Y$ in a point C whose abscissa x is such that $a = bx$. Moreover, there is no other number y such that $a = by$. This can also be seen as follows:

If there were another number y such that $by = a$, we should have $bx = by$. But it follows from the law of cancellation for multiplication that this relation can be true only when $x = y$. Hence the number x is the only number that satisfies the condition of the theorem.

Definition of division. — The process of finding x when a and b are given is called the process of **dividing** a *by* b. In this process a is called the **dividend**, b the **divisor**, and x the **quotient.**

Division is indicated by writing the dividend over the divisor with a horizontal line between them, and also by the symbol $\div$.

Thus, $\frac{a}{b}$ indicates the division of a by b, as does also the symbol $a \div b$. The relation $a = bx$ is then equivalent to the relation $\frac{a}{b} = x$, provided that b is not 0.

9. *Multiplication is distributive as to subtraction.*

For if $a - b = x$,

then $a = x + b$, (Definition of Subtraction.)

and $ma = mx + mb$. (XIII)

Hence, $ma - mb = mx = m(a - b)$.

(Definition of Subtraction and VII.)

10. *The product of* 0 *and any number is* 0, *and conversely, if the product of two numbers is* 0, *one of the numbers must be* 0.

This can be seen directly from the figure.

In the definition of division it was stipulated that the divisor should not be 0. The reason for this is clear from Theorem 10. Since $x \cdot 0 = 0$ for all values of x, there can be no value of x for which $x \cdot 0 = a$, where a is different from 0. Hence $\frac{a}{0}$ does not represent any number when a is not 0. On the other hand, $\frac{0}{0}$ might, with our definition of division, consistently stand for any number.

The student should therefore be careful never to attempt to divide by 0 because there is no such operation as division by 0.

It follows from Theorem 10 that $\frac{0}{b} = 0$, for all values of b different from 0.

11. *The product of a positive number by a negative number is a negative number.*

Let a and b be any positive numbers.

Then there is a number x such that

$$a(-b) = x, \quad \text{(VII)}$$

and therefore $a(-b) + ab = x + ab$. (I)

$$a[(-b) + b] = x + ab. \quad \text{(XIII)}$$

$$a \cdot 0 = x + ab. \quad \text{(VI)}$$

$$0 = x + ab. \quad (10)$$

$$0 - ab = x. \quad \text{(Definition of Subtraction.)}$$

$$0 + (-ab) = x. \qquad (3)$$

$$(-ab) = x. \qquad (1)$$

Hence, $a(-b) = (-ab)$.

But ab is a positive number (XIV) and therefore $(-ab)$ is negative.

12. *The product of two negative numbers is a positive number.*

Let $(-a)$ and $(-b)$ be any negative numbers.

Then there is a number x such that

$$(-a)(-b) = x, \qquad \text{(VII)}$$

and therefore $(-a)(-b) + a(-b) = x + a(-b)$.

$$(-b)[(-a) + a] = x - ab.$$

$$(-b) \cdot 0 = x - ab.$$

$$0 = x - ab.$$

$$0 + ab = x.$$

$$ab = x. \qquad (1)$$

Therefore $(-a)(-b) = ab$.

Moreover ab is a positive number. (XIV)

The well-known rule of signs for multiplication follows from XIV, 11, and 12.

13. *The result of dividing a positive number by a positive number, or a negative number by a negative number, is a positive number.*

14. *The result of dividing a positive number by a negative number, or a negative number by a positive number, is a negative number.*

The proofs of 13 and 14 depend upon XIV, 11, 12, and the definition of division. The details are left as exercises for the student.

11. The removal and insertion of parentheses.— The rule for removing single parentheses is contained in Theorems 5 and 6. Two or more parentheses, no one of which is contained within another one, can be removed simultaneously by the same rule. When parentheses are contained within parentheses, it is best

to proceed step by step, at each step removing the innermost parentheses. In such cases more than one pair of parentheses can be removed at a time if care is exercised to get the signs correct.

If the expression in parentheses is to be multiplied by a monomial factor, every term in the parentheses should be multiplied by this factor. This follows from Assumption XIII.

The rule for the insertion of parentheses is an immediate consequence of the rule for their removal. We leave the formulation of it as an exercise for the student.

EXERCISES

Simplify the following expressions:

1. $2-[3+5-7\{3-(5+2)-1\}+1]$.
2. $1+2\{3-5+2(-6+2)\}$.
3. $4a-3-(x-3a)+(2a-x)$.
4. $2x^2y-3(x^3+y^3)+2(xy^2-x^2y)$.
5. $6x-2[3x+4-3(x-1)]$.
6. $8m-[-(m-n)+(3n-6m)]$.
7. $6b-5(b-2)+6[b-(5+3b)+7]$.
8. $4a-(-3a-[-4a-\{-2a-\overline{3a+1}\}-10a])$.
9. $7x+2[3x-\{2x-(y-2x)\}-2(x-3y)]$.
10. $a-(-a-\{-7a-3[a-\overline{3a-4}]+3\}-[3a-\overline{a+1}])$.
11. $a(x+y)-b(3x-2y)$.
12. $2a(a^3-b^3)-b[ab^2-3(4a^2-5b^2)]$.
13. $x-[x-\{x-(x-1)\}]$.
14. $3m-2n[1+3m(1-\overline{m+n})]$.
15. $2(x+y+z)-3(x-y+z)+2(x+y-z)$.
16. $5ab-3b\{a+b-2(3a-2b)\}$.
17. $a(x+y)+b(x-y)-x(a+b)+y(a-b)$.

18. $(a+b-c)+(a-b+c)+(-a+b+c)$.

19. $4m+2\{n-3(m-n)\}-\overline{3m-2n}$.

20. $\overline{3m-2n}-2\{n-3(m-n)\}-4m$.

In the following polynomials inclose all the terms containing x^2 in parentheses preceded by the plus sign and inclose the terms containing x in parentheses preceded by the minus sign:

21. $4x^2+ax^2-3x+bx+5$.

22. $x^2+2ax+mx^2-2x+3+kx$.

23. $bx+x^2+cx+1$.

24. $2a^2b^2-2b^2x-a^2x^2+x^2$.

25. $-ax+x^2-bx-2x-5$.

26. $7x-6x^2+4a-3bx-cx^2$.

27. $ax^2+bx+c-3x^2-10x+4-bx^2$.

28. $3+a+x+x^2-mx-nx^2-vx$.

29. $-3x+5-ax^2+bx^2+cx^2-ax-bx-cx$.

30. $3(ax^2-2bx+c)-5(a+cx-bx^2)$.

12. Addition and subtraction of polynomials. — The problem of expressing the sum of the polynomials $3x-4y+6z-2$, $-5x-6y+z$, and $3x+2y+z+5$ in the simplest possible form is the same as the problem of simplifying the expression

$$(3x-4y+6z-2)+(-5x-6y+z)+(3x+2y+z+5).$$

Now this is equal to

$$\begin{aligned}&3x-4y+6z-2-5x-6y+z+3x+2y+z+5\\&=3x-5x+3x-4y-6y+2y+6z+z+z-2+5\\&=x-8y+8z+3.\end{aligned}$$

The usual rule for finding the sum of these polynomials is merely a more direct way to this same result. And in general the rule for finding the sum of any polynomials is a direct

consequence of Theorem 5 and Assumption II. The rule for simplifying the sum of two or more similar terms is a consequence of Assumption XIII.

We get the rule for subtracting one polynomial from another in a similar way by using Theorem 6 instead of Theorem 5.

EXERCISES

1. Add $7a^3 - 4a + 2a^2 - 5$, $-2a + 5a^3 + 1 - 2a^2$, and $4 + 3a + 2a^2 + a^3$.

2. Add $-5x^2y - 2xy^2 - x^3 - y^3$, $x^3 + y^3$, $x^3 - y^3$, and $x^3 + 3x^2y + 3xy^2 + y^3$.

3. Subtract $3a^3 + 8a^2b - 5ab^2 + 2b^3$ from $a^3 - 6a^2b + 4ab^2 + 3b^3$, and then add the subtrahend to the result. Can you predict what the final result will be?

4. Subtract $x^3 - y^3$ from $x^3 + y^3$.

5. Subtract $x^3 + y^3$ from $x^3 - y^3$.

6. From the sum of $y^2 - 8y + 2$, $7y - 2 + y^2$, and $6y^2 + 6y + 6$ subtract $y^2 + 5y + 6$.

7. From the sum of $4a^2b + 5ab^2$, $5a^3 - 2ab^2 + 3b^3$, and $2a^3 - 2b^3 + 3a^2b + 7ab^2$ subtract $7a^3 + 7a^2b + 10ab^2 + b$.

8. Add $(r+s+t)x + (r-s+t)y + (r+s-t)z$, $(2r-s)x + (s+3t)y + (r+s+t)z$, and $(s+3t)x + (2r+4t)y + (s+3t)z$.

9. From the sum of $mx^3 + 3nx^2 + px - q$ and $nx^3 - mx^2 + 2qx + p$ subtract $(m+n)x^3 + (m-3n)x^2 + (p+q)x + (q-p)$.

10. Subtract $(a+b)x + (b+c)y + (c+a)z$ from $(c+a)x + (a+b)y + (b+c)z$.

11. Subtract $(c+a)x + (a+b)y + (b+c)z$ from $(a+b)x + (b+c)y + (c+a)z$.

12. Does a consideration of the results in Exs. 10 and 11 suggest to you any general principle? Consider also Exs. 4 and 5.

13. Add $a+b-c$, $a-b+c$, and $-a+b+c$. (See Ex. 18, § 11).

14. Subtract $2(a+b-c)$ from $a+b+c$.

15. From the sum of $a^3+3a^2b+3ab^2+b^3$ and $a^3-3a^2b+3ab^2-b^3$ subtract the sum of a^3+b^3 and a^3-b^3.

16. Subtract a^3+b^3 from $a^3+3a^2b+3ab^2+b^3$, and a^3-b^3 from $a^3-3a^2b+3ab^2-b^3$.

17. Is there any difference between the result in Ex. 15 and the sum of the results in Ex. 16? Could you have answered this question without going through the details of Exs. 15 and 16?

18. Subtract ax^2+bx+c from dx^2+ex+f.

19. Is the result of subtracting a number from zero the same as that of subtracting zero from this number?

20. What must be added to $7a^3-8a^2+4a+7$ in order that the sum shall be $2a^3+4a^2+6a-1$?

21. What must be added to $4x^2y-6xy^2+8y^3$ in order that the sum shall be $5x^3+7xy^2+y^3$?

13. Multiplication of polynomials. — The rule for forming the product of two polynomials is a direct consequence of Assumption XIII.

Consider the product of the two binomials $a+b$ and $x+y$. By Assumption XIII

$$(a+b)(x+y)=(a+b)x+(a+b)y=ax+bx+ay+by.$$

The product of any two polynomials can be treated step by step in a similar way.

Homogeneous polynomials. — A polynomial all of whose terms are of the same degree is said to be **homogeneous.**

Thus, $5x^2+7y^2$ and $a^3+3a^2b+3ab^2+b^3$ are homogeneous polynomials of degrees 2 and 3 respectively.

The product of two homogeneous polynomials is a homogeneous polynomial whose degree is equal to the sum of the degrees of the factors. This fact is useful in checking the work of forming the product of two homogeneous polynomials.

EXERCISES

Simplify the following indicated products and use the foregoing check whenever it is applicable:

1. $(x+2y)(x+3y)$.
2. $(a+5)(a-8)$.
3. $(x^2+2xy+y^2)(x^2-2xy+y^2)$.
4. $(x^2-2xy+y^2)(x^3-3x^2y+3xy^2-y^3)$.
5. $(a^2+ab+b^2)(a^2-ab+b^2)$.
6. $(a+ar+ar^2+ar^3+ar^4)(1-r)$.
7. $(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.
8. $(x+y+z)(x+y-z)$.
9. $(a+b+c)(2a-b+3c)$.
10. $(a^2+6a+4)(a^3-4a^2+5a+2)$.
11. $(a+b+x-y)(a-b-x+y)$.
12. $(3a^3-5a^2b+7ab^2+4b^3)(4a^2+10ab+5b^2)$.
13. $(x^2+y^2+z^2+2xy+2yz+2zx)(x+y+z)$.
14. $(2a-3b+5)(2a-3b-4)$.
15. $(m^4+m^2n^2+n^4)(m^4-m^2n^2+n^4)$.
16. $(c^3+9c^2+27c+27)(c^2-6c+9)$.
17. $(16-8x+4x^2-2x^3+x^4)(2+x)$.
18. $(r^2+5r-4)(r^2+5r-3)$.
19. $(7a^2+5b^2-9c^2)(6a^2-4b^2+c^2)$.
20. $(x^2+3x+9)(x^2-3x+9)$.

14. Special products. — The following special products should be memorized by the student in order that he may be able to write down products of these types from an inspection of the factors. He should verify each of the formulæ by actual multiplication. He should also translate each one into ordinary language.

1. $(a+b)^2 = a^2 + 2ab + b^2.$
2. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$
3. $(a+b)(a-b) = a^2 - b^2.$
4. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$

EXERCISES

Write the results of the following indicated multiplications:

1. $(a-b)^2.$
2. $(a-b)^3.$
3. $(2x+3y)^2.$
4. $(2x+3y)^3.$
5. $(2x+3y)(2x-3y).$
6. $(3a-b)^2.$
7. $(a-b-c)^2.$
8. $(2x-3y+4z)^2.$
9. $(a+b+c)(a+b-c).$
10. $(2m-5n)(2m-5n).$
11. $(x+2)^3.$
12. $(5a-4)^2.$
13. $(10+9b)(10-9b).$
14. $(a+1)^3.$
15. $(7x+10y)(10y-7x).$
16. $(3m+n+r)^2.$
17. $(x^2+x+1).$
18. $(3r-s)^3.$
19. $(s-3r)^3.$
20. $(x+y-z)(x-y+z).$

15. Division of polynomials. — If A and B are two polynomials such that the degree of A in some letter is equal to, or greater than, the degree of B in this letter, the process of finding two polynomials Q and R such that

$$A = BQ + R,$$

and R is of lower degree in the given letter than B, is called the process of **dividing** A by B. In this process A is called the **dividend**, B the **divisor**, Q the **quotient**, and R the **re-**

mainder. If $R = 0$ we say that A is **divisible** by B, or that the division is **exact.** If the divisor is a monomial, it follows immediately from Assumption XIII that the quotient is the sum of the partial quotients obtained by dividing each term of the dividend by the divisor. How the quotient is obtained when the divisor contains more than one term is best shown by an example.

Divide $12x^3 - 7x^2y - 14xy^2 + 5y^3$ by $4x - 5y$.

$$\begin{array}{r|l} 12x^3 - 7x^2y - 14xy^2 + 5y^3 & 4x - 5y \\ \underline{12x^3 - 15x^2y} & \\ 8x^2y - 14xy^2 & 3x^2 + 2xy - y^2 \\ \underline{8x^2y - 10xy^2} & \\ -4xy^2 + 5y^3 & \\ \underline{-4xy^2 + 5y^3} & \end{array}$$

We know that the required quotient is $3x^2 + 2xy - y^2$ and that the remainder is 0, since the products of the successive terms of the quotient and the divisor when subtracted in turn from the dividend give a final remainder of 0.

Before beginning this work of finding the successive terms of the quotient, the student should see to it that the dividend and the divisor are arranged according to the descending powers of some letter, although when the division is exact they may be arranged according to the ascending powers.

The relation $$A = BQ + R,$$

holds true for all values of the letters involved and is for this reason called an **identity.** (See § 31.)

EXERCISES

Perform the following indicated divisions:

1. $(a^2 - b^2) \div (a - b)$.
2. $(x^4 + x^2y^2 + y^4) \div (x^2 + xy + y^2)$.
3. $(x^3 - y^3) \div (x - y)$.
4. $(m^6 + n^6) \div (m^2 + n^2)$.
5. $(a^4 + 4a^3 + 6a^2 + 4a + 1) \div (a^2 + 2a + 1)$.

6. $(a^4+3\,a^3+7\,a^2+8\,a+6)\div(a^2+a+3)$.

7. $(22\,x^2+13\,x^3+12+19\,x+6\,x^4)\div(5\,x+3\,x^2+4)$.

8. $(r^4-s^4)\div(r+s)$.

9. $(a^8-b^8)\div(a+b)$.

10. $[(a-b)^2-5(a-b)+6]\div[(a-b)-3]$.

11. $(m^3+n^3)\div(m-n)$.

12. $(x^2+5\,x+6)\div(x+2)$.

13. $(a^5-b^5)\div(a-b)$.

14. $(a^5-b^5)\div(a+b)$.

15. $(b^4+9\,b^2+81)\div(b^2+3\,b+9)$.

16. $(x^2+5\,x+4)\div(x^2+3\,x+2)$.

17. $(a^3+9\,a^2b+27\,ab^2+27\,b^3)\div(a^2+6\,ab+9\,b^2)$.

18. $(a^3+9\,a^2b+27\,ab^2+27\,b^3)\div(a+3\,b)$.

19. $(x^4-y^4)\div(x-y)$.

20. $(x^4-y^4)\div(x+y)$.

CHAPTER II

FACTORS AND MULTIPLES

16. The problem of factoring a polynomial — that is, of finding two polynomials whose product is the given polynomial — is, in general, more difficult than that of dividing one polynomial by another one, since in factoring both the divisor and the quotient are unknown. There are, however, certain types of polynomials that can be factored readily. The more common of these and their factors are given here.

17. 1. **Polynomials with monomial factors.** — It follows from the distributive law for multiplication that a factor of all the terms of a polynomial is a factor of the polynomial.

For example, $2x^3y^2$ is a factor of every term of the polynomial $6x^3y^2 + 8x^4y^3 - 10x^5y^3$, and is therefore a factor of the polynomial.

$$6x^3y^2 + 8x^4y^3 - 10x^5y^3 = 2x^3y^2(3 + 4xy - 5x^2y).$$

2. **Polynomials that can be factored by grouping their terms.** — The following is a typical polynomial of this form:

$$\begin{aligned} ax + ay + az + bx + by + bz &= (ax + ay + az) + (bx + by + bz) \\ &= a(x + y + z) + b(x + y + z) \\ &= (x + y + z)(a + b). \end{aligned}$$

3. **The difference of two squares.**

$$a^2 - b^2 = (a + b)(a - b).$$

Many polynomials that are more complicated in appearance than this one are in reality of this form. For example,

$$\begin{aligned} a^2 + 2ab + b^2 - x^2 + 2xy - y^2 &= (a^2 + 2ab + b^2) - (x^2 - 2xy + y^2) \\ &= (a + b)^2 - (x - y)^2 \\ &= (a + b + x - y)(a + b - x + y). \end{aligned}$$

4. **The sum of two cubes.**

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

5. **The difference of two cubes.**

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

6. **Trinomial squares.**

$$a^2 + 2ab + b^2 = (a + b)^2.$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

7. **The quadratic trinomial $ax^2 + bx + c$.**

If four numbers k, l, m, and n can be found such that $km = a$, $lm + kn = b$, and $ln = c$, then $ax^2 + bx + c = (kx + l)(mx + n)$.

8. **Polynomials of four terms that are the cubes of binomials.**

$$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3.$$
$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3.$$

9. **The sum of two like odd powers.**

If n is an odd positive interger $a^n + b^n$ is divisible by $a + b$.

10. **The difference of two like powers.**

For every positive integral value of n, $a^n - b^n$ is divisible by $a - b$.

This statement and the one under 9 can be proved by means of the factor theorem (§ 142, Cor.)

18. Whether a given polynomial can be factored or not depends somewhat upon our point of view. Thus, the polynomial $x^2 - 2$ has the factors $x + \sqrt{2}$ and $x - \sqrt{2}$ although it has none with rational numerical coefficients. Also $x^2 - y$ has the factors $x + \sqrt{y}$ and $x - \sqrt{y}$, but none that are free from radicals.

In this chapter we shall confine our attention to polynomials whose numerical coefficients are integers, and when we speak of the factors of a polynomial we shall mean factors that are free from radicals and that have integral numerical coefficients.

Definition. — A polynomial that has no factors of this kind is said to be **prime.**

A polynomial whose numerical coefficients are integers with a common factor greater than 1 is not prime since this numerical factor is a factor of the kind just described.

The work of factoring can be checked by making use of the fact that the product of all the factors found should equal the given polynomial.

EXERCISES

Factor the following polynomials into their prime factors:

1. $12\,a^5b^4 + 18\,a^4b^5 - 21\,a^3b^6$.
2. $(a+b)^2 - 1$.
3. $8\,x^3 + 12\,x^2 + 6\,x + 1$.
4. $9\,a^2 + 24\,abc + 16\,b^2c^2$.
5. $27 - 64\,x^3$.
6. $49 - x^2 + 4\,xy - 4\,y^2$.
7. $a^2 + 4\,a - 77$.
8. $a^2 - b^2 - a + b$.
9. $a^6 - 2\,a^3 + 1$.
10. $x^3 + x^2 + x + 1$.
11. $a^7b - ab^7$.
12. $2\,xy - x^2 - y^2$.
13. $x^3 - 3\,x^2y + 3\,xy^2 - y^3 - z^3$.
14. $12\,a^2 + 13\,ab - 35\,b^2$.
15. $6\,x^2 + 7\,xy - 20\,y^2$.
16. $a^2 - 5\,a + 4$.
17. $m^4 - 25\,m^2 - m + 5$.
18. $-2\,x^2 + 3\,xy + 27\,y^2$.
19. $x^4 + 34\,x^2y^2 + 225\,y^4$.
20. $a^3 + 4\,a^2 + 2\,a + 8$.
21. $54\,a^3x^2 - 16\,b^3x^2$.
22. $6\,x^2 - x - 15$.
23. $a^4 - x^4$.
24. $m^2 + 2\,xy - x^2 - y^2$.
25. $x^5 - y^5$.
26. $x^5 + y^5$.
27. $15\,a^2 + 47\,ab + 28\,b^2$.
28. $(x+y)^3 - 27$.
29. $6\,x^3 + 3\,x^2 + 2\,x + 1$.
30. $a^3 + 6\,a^2 + 12\,a + 8$.

19. Highest common factor. — A polynomial that is a factor of each of two or more polynomials is called a **common factor** of these polynomials. A given set of polynomials may have several common factors of the same degree.

Thus, $2\,x^2 - 4\,xy + 2\,y^2$ and $4\,x^2 - 4\,y^2$ have the common factors $x - y$, $-x + y$, $2\,x - 2\,y$, and $-2\,x + 2\,y$ of the first degree.

Among all the common factors of the highest degree of two or more polynomials there is always one the greatest common divisor of whose numerical coefficients is equal to the greatest common divisor of all the numerical coefficients of the given polynomials. We call this a **highest common factor** of the polynomials.

Thus, $2x - 2y$ is a highest common factor of the polynomials given in the preceding illustration.

But $-2x + 2y$ is also a highest common factor of these polynomials; and in general, any set of polynomials has two highest common factors which differ only in sign. In most cases it makes no difference which of these two highest common factors of the given polynomials we take, and it is therefore customary to speak of "the highest common factor," as if there were but one.

If the given polynomials can be factored into their prime factors, their H. C. F. (highest common factor) can be determined immediately by inspecting these prime factors.

20. Lowest common multiple. — A polynomial that is divisible by each of two or more polynomials is called a **common multiple** of these polynomials. A given set of polynomials has many common multiples of the same degree.

Thus, $2x^2 - 4xy + 2y^2$ and $4x^2 - 4y^2$ have the common multiples $4x^3 - 4x^2y - 4xy^2 + 4y^3$, $8x^3 - 8x^2y - 8xy^2 + 8y^3$, and many others which are of the third degree.

Among all the common multiples of the lowest degree of two or more polynomials there is always one the greatest common divisor of whose numerical coefficients is equal to the least common multiple of the greatest common divisors of the numerical coefficients of the respective polynomials. We call this a **lowest common multiple** of the polynomials.

Thus, $4x^3 - 4x^2y - 4xy^2 + 4y^3$, is a lowest common multiple of the two polynomials given in the preceding illustration.

Every set of polynomials has two lowest common multiples which differ only in sign. The expression "the lowest common multiple" refers to either one.

If the given polynomials can be factored, their L. C. M. (lowest common multiple) can readily be formed.

EXERCISES

Find the H. C. F. and the L. C. M. of the following polynomials:

1. $25, 30, 40.$
2. $x^3 - y^3,\ x^2 - y^2.$
3. $a^2 + 2\,ab + b^2,\ a^2 + 3\,ab + 2\,b^2.$
4. $b^3 + y^3,\ b^3 - y^3.$
5. $x^2 + y^2,\ x^4 - y^4,\ x^6 - y^6.$
6. $x^5 - 243,\ x^2 - 6\,x + 9,\ x^2 - 9.$
7. $ax - bx - ay + by,\ a^2 + ab - 2\,b^2.$
8. $a^2 + 3\,a - 28,\ 5\,a^2 - 20\,a.$
9. $a^6 - x^6,\ a^3 - x^3,\ a^2 - x^2.$
10. $x^2 + x - 6,\ 2\,x^2 + 2\,x - 12.$
11. $a^2 - b^2,\ b^2 - a^2.$
12. $x^2 + 7\,x + 12,\ x^2 + 5\,x + 6.$
13. $6\,a^2 - 13\,ab + 6\,b^2,\ 10\,a^2 - 13\,ab - 3\,b^2.$
14. $x^2 + y^2,\ x^2 - y^2.$
15. $24\,m^2 - 14\,mn - 20\,n^2,\ 24\,m^2 + 6\,mn - 45\,n^2.$
16. $16\,a^4 - 81\,b^4,\ 8\,a^3 - 27\,b^3.$
17. $x^2 + 6\,xy + 9\,y^2,\ x^3 + 9\,x^2y + 27\,xy^2 + 27\,y^3.$
18. $ax^3 + 2\,ax^2 + 3\,ax + 6\,a,\ 4\,ax + 8\,a.$
19. $15\,a^2 + 11\,ab + 2\,b^2,\ 5\,a^2 + 7\,ab + 2\,b^2.$
20. $a + b,\ a + 2\,b,\ a + 3\,b.$

21. Euclid's Method for finding the highest common factor of two or more polynomials. — If we cannot find the factors of the given polynomials, the preceding method cannot be used. In such a case, if the polynomials contain only one letter, we can fall back on the following method, which is called *Euclid's Method.*

Let A and B be any two polynomials in x whose H. C. F. we wish to find. Arrange them according to the descending powers of x, and if the degree of B is not greater than the degree of A divide A by B.

Denote the quotient by Q_1 and the remainder by R_1. Then (§ 15)

$$A = BQ_1 + R_1,$$

or

$$A - BQ_1 = R_1.$$

Now any factor of B is a factor of BQ_1. Hence any common factor of A and B is a factor of $A - BQ_1$, which is the same as R_1; and is therefore a common factor of B and R_1. Conversely, any common factor of B and R_1 is a factor of $BQ_1 + R_1$, which is the same as A; and is therefore a common factor of A and B. The highest common factor of A and B must accordingly be the same as the highest common factor of B and R_1. We can then shift the problem to the finding of the highest common factor of B and R_1. Now R_1 is of lower degree in x than B. We therefore proceed as before and divide B by R_1. If the quotient is Q_2 and the remainder R_2, we have

$$B = R_1Q_2 + R_2,$$

or

$$B - R_1Q_2 = R_2.$$

Reasoning similar to that used before shows that the highest common factor of B and R_1 is the highest common factor of R_1 and R_2.

By continuing in this way we can establish the following series of identities :

$$A = BQ_1 + R_1,$$
$$B = R_1Q_2 + R_2,$$
$$R_1 = R_2Q_3 + R_3,$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$R_k = R_{k+1}Q_{k+2} + R_{k+2},$$

in which R_{k+2} does not contain x. This is possible since each R is of lower degree in x than the preceding one, and if the process is continued far enough we must come to an R that does not contain x.

It is clear from the method of formation of these equations that the highest common factor of R_k and R_{k+1} is also the highest common factor of A and B. If now $R_{k+2} = 0$, R_{k+1} is the highest common factor of R_k and R_{k+1} and therefore also of A and B. But if $R_{k+2} \neq 0$, A and B have no common factor containing x.

Example. — Find the H. C. F. of $x^3 + 2x^2 - 13x + 10$ and $x^3 + x^2 - 10x + 8$.

These two polynomials are of the same degree and either may therefore be taken as the divisor. We select the second one.

$$\begin{array}{l|l} x^3 + 2x^2 - 13x + 10 & x^3 + x^2 - 10x + 8 \\ \underline{x^3 + x^2 - 10x + 8} & 1 \\ x^2 - 3x + 2 & x^3 + x^2 - 10x + 8 \\ \underline{x + 4} & \underline{x^3 - 3x^2 + 2x} \\ & 4x^2 - 12x + 8 \\ & \underline{4x^2 - 12x + 8} \end{array}$$

Here $R_2 = 0$ and therefore $R_1 = x^2 - 3x + 2$ is the H. C. F.

In applying this process the given polynomials should first be divided by any monomial factors they may have and account taken of these factors in the final result. Fractional coefficients should be avoided by introducing or removing numerical factors whenever necessary at any stage of the process. This can always be done in such a way as not to affect the final result.

22. Lowest common multiple of two polynomials. — If F is the highest common factor of the polynomials A and B, and $A = A'F$, $B = B'F$, then A' and B' have no common factor. Now $\frac{AB}{F} = AB' = A'B$. Hence $\frac{AB}{F}$ is a common multiple of A and B. Suppose now that M is any common multiple of A and B. Then we should have $M = AQ = A'FQ$. But M must also be divisible by B, which is the same as $B'F$. Hence Q must be divisible by B'. In order that M be the lowest common multiple, Q must be the same as B'. Hence the L. C. M. of A and B is $A'FB' = AB' = \frac{AB}{F}$. This is equivalent to saying that

The lowest common multiple of two polynomials is equal to their product divided by their highest common factor.

EXERCISES

Find the H. C. F. and the L. C. M. of the following polynomials:

1. $x^3 - x^2 + 3x + 5$, and $x^3 + 3x^2 - 3x - 5$.
2. $x^4 + 4x^3 + 5x^2 + 4x + 1$ and $x^4 + 5x^3 + 3x^2 - 10x - 4$.
3. $6x^3 - 22x^2 + 15x - 2$ and $6x^4 - x^3 + 4x^2 - 13x + 6$.
4. $2x^3 + 12x^2 + 19x + 3$ and $2x^3 + 13x^2 + 17x + 3$.
5. $x^4 + 3x^3 + 6x^2 + 3x - 5$ and $x^4 + x^3 - 11x^2 - 10x + 10$.
6. $x^3 + 6x^2 + 11x + 6$ and $x^3 + 8x^2 + 11x - 20$.
7. $2a^4 + 4a^3 - 2a^2 - 4a$ and $a^3 + 2a^2 - 3$.
8. $y^4 + 3y^3 + 4y^2 - 3y - 5$ and $y^4 + 3y^3 + 6y^2 + 3y + 5$.
9. $a^3 + 3a^2 + 8a + 6$ and $a^4 + 3a^3 + 9a^2 + 8a + 6$.
10. $b^2 - 6b + 2$ and $b^3 - 2b^2 - 22b + 8$.
11. $x^5 + 4x^3 + x^2 - 5x + 1$ and $x^4 + x^3 + 6x^2 + 5x + 5$.

CHAPTER III

FRACTIONS

23. Definition and principles. — A fraction is the indicated quotient of two numbers.

Thus, $\frac{2}{3}, \frac{-3}{5}, \frac{-1.4}{-.5}, \frac{\sqrt{2}}{3}$.

The principles upon which the usual operations with fractions rest can be readily derived from the assumptions of Chapter I.

These principles are:

1. *The value of a fraction is not changed by multiplying the numerator and the denominator by the same number.*

Consider the fraction $\frac{a}{b}$. It follows from the definition of division that

$$a = b \cdot \frac{a}{b}.$$

Hence,

$$ak = bk \cdot \frac{a}{b}.$$

But this is only another way of saying that

$$\frac{ak}{bk} = \frac{a}{b}.$$

This holds for all values of k except 0, and since division by a number is equivalent to multiplication by the reciprocal of this number, we have also proved that *the value of a fraction is not changed by dividing the numerator and the denominator by the same number.*

2. *The sign of a fraction is changed if the sign of either the numerator or the denominator is changed.*

This is an immediate consequence of the law of signs for division, which is derived from the assumptions and the definition of division.

3. *The sum of two fractions with a common denominator is a fraction whose numerator is the sum of the numerators of the given fractions and whose denominator is the common denominator.*

Let the given fractions be $\frac{a}{c}$ and $\frac{b}{c}$.

Now $$a = c \cdot \frac{a}{c},$$

and $$b = c \cdot \frac{b}{c}.$$ (Definition of division.)

Hence, $$a + b = c\frac{a}{c} + c\frac{b}{c}$$

$$= c\left(\frac{a}{c} + \frac{b}{c}\right).$$ (Distributive law for multiplication.)

From this it follows that

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$$ (Definition of division.)

In a similar way we see that

$$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}.$$

In view of what has just been proved we can say that *division is distributive as to addition.*

This includes the statement that division is distributive as to subtraction since subtraction is included under addition. (See 3, Chapter I.)

4. *The sum and the difference of any two fractions,* $\frac{a}{b}$ *and* $\frac{c}{d}$, *are equal to* $\frac{ad+bc}{bd}$ *and* $\frac{ad-bc}{bd}$ *respectively.*

For by 1, $$\frac{a}{b} = \frac{ad}{bd},$$

and $$\frac{c}{d} = \frac{bc}{bd}.$$

Hence by 3, $$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd},$$

and $$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

5. *The product of two fractions is a fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.*

For, $$b \cdot \frac{a}{b} \cdot \frac{c}{d} = a \cdot \frac{c}{d}.$$

(Definition of division, and IX, Chapter I.)

and $$bd \cdot \frac{a}{b} \cdot \frac{c}{d} = b \cdot \frac{a}{b} \cdot \frac{c}{d} \cdot d = a \cdot \frac{c}{d} \cdot d = ac.$$

(Definition of division, VIII, and IX, Chapter I.)

Hence, $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$ (Definition of division.)

6. *The quotient resulting from dividing one fraction by another is a fraction whose numerator is the product of the numerator of the dividend and the denominator of the divisor and whose denominator is the product of the denominator of the dividend and the numerator of the divisor.*

If the dividend is $\frac{a}{b}$ and the divisor $\frac{c}{d}$, we have, from the definition of division,

$$\frac{a}{b} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{c}{d}.$$

Hence, $$ad = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot bc,$$

and $$\frac{ad}{bc} = \frac{\frac{a}{b}}{\frac{c}{d}}.$$

When we are dealing with more than two fractions the principles to be employed are obvious extensions of those just developed.

The following reductions are applicable to **rational** fractions; that is, fractions which are equivalent to indicated quotients of polynomials.

24. Reduction of certain fractions to mixed expressions. — If the degree of the numerator of a fraction in a certain letter is equal to, or greater than, the degree of the denominator in this letter, we can arrange both numerator and denominator according to the descending powers of this letter and divide the numerator by the denominator. Then, if the given fraction is $\frac{a}{b}$, we have the identity

$$\frac{a}{b} = q + \frac{r}{b},$$

where q is a quotient, and r the remainder, of this division. Here q is a polynomial and $\frac{r}{b}$ is a fraction whose numerator is of lower degree than its denominator. Hence, *if the degree of the numerator of a fraction in any letter is equal to, or greater than, the degree of the denominator in this letter, the fraction is equal to the sum of a polynomial and a fraction whose numerator is of lower degree in this letter than its denominator.*

25. Reduction of a fraction to its lowest terms. — A fraction is said to be in its lowest terms when its numerator and denominator have no common factor.

A fraction can be reduced to its lowest terms by dividing the numerator and denominator by their highest common factor (§ 23, 1).

26. Reduction of several fractions to equivalent fractions with a common denominator of lowest possible degree. — Find the L. C. M. of the denominators, which is called the **lowest common denominator** of the fractions, and divide this by the denominator of each fraction. Then multiply the numerator and the denominator of each fraction by the corresponding quotient. The resulting fractions are equivalent to the orig-

inal ones (§ 23, 1) and have the required common denominator. It is unnecessary actually to perform the multiplication of each denominator by the corresponding quotient, since we know beforehand that the product will be the lowest common denominator.

27. Addition and subtraction of fractions. — In order to add or subtract fractions, first reduce them to equivalent fractions with a common denominator and then proceed in accordance with § 23, 3.

In order to add a polynomial to a fraction, consider the polynomial as a fraction with the denominator 1 and then proceed as directed.

28. Multiplication and division of fractions. — The rules for multiplying and dividing fractions are given in § 23, 5, 6. The given fractions should be reduced to their lowest terms before applying these rules.

EXERCISES

Reduce the following fractions to mixed expressions:

1. $\dfrac{a^2+ab+b^2}{a+b}$.
2. $\dfrac{x^3-1}{x^2-1}$.
3. $\dfrac{a^3}{a-1}$.
4. $\dfrac{x^3+2x^2-x+4}{x^3-x^2+2x-4}$.
5. $\dfrac{x^4+1}{x^2-1}$.
6. $\dfrac{4x}{x+1}$.

Reduce the following fractions to their lowest terms:

7. $\dfrac{18x^2y^3}{12xy^4}$.
8. $\dfrac{x^2-9}{x^2-6x+9}$.
9. $\dfrac{x^5-y^5}{y^6-x^6}$.
10. $\dfrac{m^3+5m^2+2m+10}{m^2+7m+10}$.
11. $\dfrac{x^2-(y+z)^2}{(x+y)^2-z^2}$.
12. $\dfrac{x^3+4x^2+8x+5}{x^3-3x^2+3x+7}$.

Perform the following indicated operations and bring your results to their simplest forms:

13. $\dfrac{1}{x+1}+\dfrac{1}{x-1}$.

14. $\dfrac{4}{x+1}-\dfrac{2}{x-1}+\dfrac{3}{x+3}$.

15. $\dfrac{3}{a^2-2a+1}-\dfrac{4a}{a^2-1}$.

16. $\dfrac{x+2}{x^2+5x+6}+\dfrac{x+3}{x^2+7x+12}$.

17. $\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(x-3)}+\dfrac{1}{(x-3)(x-1)}$.

18. $m^2-m+1-\dfrac{m^3}{m+1}$.

19. $x+\dfrac{1}{x+1}$.

20. $\dfrac{3}{a+b}-\dfrac{2}{(a+b)^2}+\dfrac{a-b}{(a+b)^3}$.

21. $\dfrac{2a^2b^3}{5x^3y^4}\cdot\dfrac{10axy}{3b^2}$.

22. $\left(\dfrac{7x^3y^2z}{9ab}\right)^3$.

23. $(a^2-6a+9)\left(\dfrac{3a+2}{a-3}\right)^2$.

24. $\left(\dfrac{1}{a}-\dfrac{1}{b}\right)\left(\dfrac{1}{a}+\dfrac{1}{b}\right)\left(1+\dfrac{a+b}{a-b}\right)\left(2+\dfrac{2b}{a-b}\right)$.

25. $\dfrac{m^2+mn+mr+nr}{m^2-mn-mr+nr}\div\dfrac{m^2-r^2}{m^2-n^2}$.

26. $\left(1+\dfrac{6}{x}+\dfrac{5}{x^2}\right)\div\dfrac{x^2+10x+25}{3x^2}$.

27. $\left(\dfrac{m^2-n^2}{m^2+n^2}-\dfrac{m^2+n^2}{m^2-n^2}\right)\div\left(\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}\right)$.

28. $\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}\right)\dfrac{1}{r_1^2-r_2^2}$.

29. $\dfrac{x^2+5x+4}{x^2+5x+6}\cdot\dfrac{x^2+6x+8}{x^2+4x+3}$.

30. $\dfrac{x^3-y^3}{3x}\cdot\dfrac{6xy}{x^2+3xy-4y^2}$.

Simplify the following:

31. $\dfrac{nE}{\dfrac{nr}{m}+R}$.

32. $\dfrac{g}{1+\dfrac{2h}{R}}$.

33. $\dfrac{H}{1+\dfrac{t}{5508}}$.

34. $\dfrac{ne}{\dfrac{nr}{N}+l}$.

35. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}$.

29. Ratio. — The fraction $\frac{a}{b}$ is called the **ratio** of a to b. The numerator a is called the **antecedent** of the ratio and the denominator b the **consequent.**

Two numbers a and b are said to be **commensurable** if there is a number r such that $\frac{a}{r}$ and $\frac{b}{r}$ are integers. If we call these integers x and y respectively, that is, if $\frac{a}{r}=x$ and $\frac{b}{r}=y$, then $\frac{a}{b}=\frac{x}{y}$. Hence the ratio of the two commensurable numbers is a rational fraction. Not all numbers are commensurable.

For example, $\sqrt{2}$ and 1 are **incommensurable**, since if they were commensurable we should have $\frac{\sqrt{2}}{1}=\frac{x}{y}$ or $2=\frac{x^2}{y^2}$. Hence, $2y^2=x^2$. But every prime factor of x^2 and of y^2 occurs to an even power and this relation requires that 2 occur as a factor of x^2 to a power one higher than the power to which it occurs in y^2.

30. Proportion. — The statement that two fractions are equal is called a **proportion.** The numerators and denominators of the fractions are called the **terms of the proportion.** Thus, a, b, c, and d are the terms of the proportion $\frac{a}{b}=\frac{c}{d}$. This proportion is sometimes read "a is to b as c is to d." The first and the last terms are called the **extremes,** and the other two the **means,** of the proportion.

If we clear the equation $\frac{a}{b}=\frac{c}{d}$

of fractions, we get $ad=bc.$

Hence, *in any proportion the product of the means is equal to the product of the extremes.*

The student can verify that if

$$\frac{a}{b}=\frac{c}{d},$$

then $$\frac{a}{c}=\frac{b}{d},$$

and $$\frac{b}{a}=\frac{d}{c}.$$

The former of these two is said to have been obtained from the original one by **alternation** and the latter by **inversion.**

Suppose that $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=r.$$

Then $$a=br,$$

$$c=dr,$$

and $$e=fr.$$

Hence, $$a+c+e=br+dr+fr=r(b+d+f),$$

and $$\frac{a+c+e}{b+d+f}=r=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

If we had started with more than three equal ratios, it is easy to see that an analogous procedure would have led to an analogous result. We conclude therefore that *in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

If $\frac{a}{b}=\frac{c}{f}$, f is called the **fourth proportional** to a and b and c.

If $\frac{a}{m}=\frac{m}{b}$, m is called a **mean proportional** between a and b.

If $\frac{a}{b}=\frac{b}{t}$, t is called the **third proportional** to a and b.

EXERCISES AND PROBLEMS

1. In the diatonic music scale the notes C, D, E, F, G, A, and B are produced respectively by c, d, e, f, g, a, and b vibrations of the air per second, where $\frac{c}{e}=\frac{g}{b}=\frac{f}{a}=\frac{4}{5}$ and $\frac{e}{g}=\frac{b}{2d}=\frac{a}{2c}=\frac{5}{6}$. Find the number of vibrations per second that produce each of these notes, it being assumed that C is produced by 256 vibrations per second.

2. If the terms of a ratio are positive and the same positive number is added to each, in what way is the value of the ratio changed?

Let the ratio be $\frac{a}{b}$ and let x be the positive number. Then compare the values of $\frac{a}{b}$ and $\frac{a+x}{b+x}$.

Theorem from Geometry. — In a right triangle the perpendicular dropped from the vertex of the right angle to the hypotenuse is a mean proportional between the segments into which it divides the hypotenuse.

3. The hypotenuse of a right triangle is 20 inches long and the perpendicular from the vertex of the right angle upon the hypotenuse is 8 inches long. Where is the foot of this perpendicular?

4. Prove that if $\frac{a}{b}=\frac{c}{d}$, then

(a) $\frac{a+b}{b}=\frac{c+d}{d}$.

(b) $\frac{a-b}{b}=\frac{c-d}{d}$.

(c) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Proportion (a) is said to have been derived from the given proportion by **composition**, (b) by **division**, and (c) by **composition and division.**

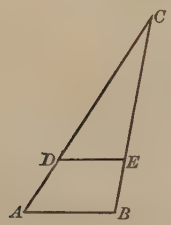
5. Find the mean proportionals between 16 and 36.

6. Find the third proportional to 15 and 28.

7. Find the fourth proportional to 13, 28, and 36.

Theorem from Geometry. — A perpendicular dropped from any point in the circumference of a circle to a diameter is a mean proportional between the segments into which it divides the diameter.

8. A perpendicular drawn from a point in the circumference of a circle of radius 6 inches to a diameter divides this diameter into two parts, the ratio of whose lengths is equal to $\frac{1}{2}$. Find the length of this perpendicular.



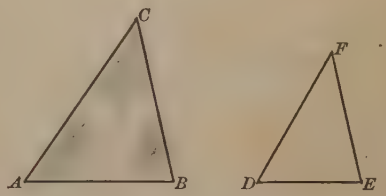
9. In the triangle ABC, $AB = 8$, $BC = 17$, and $CA = 20$. Through what point of BC must a line DE parallel to AB be drawn in order that $DE = \frac{3}{4}\,AB$?

Hint. $\dfrac{DE}{AB} = \dfrac{EC}{BC}$.

A property of similar triangles. — The areas of similar triangles are proportional to the squares of two corresponding sides.

Thus, if S_1 and S_2 are the areas respectively of the similar triangles ABC and DEF, then

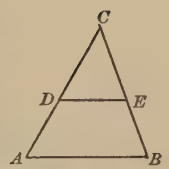
$$\frac{S_1}{S_2} = \frac{\overline{AB}^2}{\overline{DE}^2} = \frac{\overline{BC}^2}{\overline{EF}^2} = \frac{\overline{CA}^2}{\overline{FD}^2}.$$



10. If $AB = 9$ inches, what must be the length of DE in order that the triangle DEF be twice as large as the triangle ABC?

11. If $S_1 = 20$ square inches, $S_2 = 25$ square inches, and $BC = 4$ inches, how long is EF?

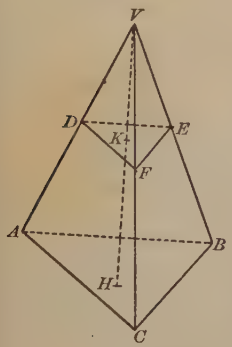
12. In the triangle ABC the line DE is to be drawn parallel to AB in such a way that the triangle cut off shall be one third of ABC. At what point in AC must D be?



Formula from Physics. — If two falling bodies pass over the distances s_1 feet and s_2 feet in t_1 seconds and t_2 seconds respectively, then $\frac{s_1}{s_2} = \frac{t_1^2}{t_2^2}$.

13. What is the ratio of the distances passed over by the two bodies if the second one falls three times as long as the first one?

14. If a body falls 64 feet in 2 seconds, how far will a body fall in 5 seconds?



Theorem from Geometry. — If a plane be drawn parallel to the base of a pyramid V–ABC, cutting the pyramid in the section DEF, and if we denote the areas of ABC and DEF by S_1 and S_2 respectively, then

$$\frac{S_1}{S_2} = \frac{\overline{VH}^2}{\overline{VK}^2},$$

where VH and VK are the distances of the base and the cutting plane respectively from the vertex.

15. In a pyramid whose altitude is 10 inches, how far from the vertex must a cutting plane be in order that the section DEF be three fourths as large as the base?

16. If the area of the base of the pyramid is 14 square inches, and the altitude is 9 inches, what is the area of the section midway between the base and the vertex?

CHAPTER IV

LINEAR EQUATIONS IN ONE UNKNOWN

31. Definition of equation. — The statement that two algebraic expressions are equal is called an **equation.**

If the statement contains only one letter whose value is unknown, it may be considered as a description of a number. A number answers this description if the two expressions have the same value when this number is substituted for the letter whose value is unknown.

An **identity** is a description that fits every number.

Thus, the two members of the identity, $(x-1)^2 = x^2 - 2x + 1$, are equal to one another regardless of what value is substituted for x.

An identity is sometimes indicated by the symbol $\equiv$.

Thus, $(x-1)^2 \equiv x^2 - 2x + 1$.

On the other hand, the two members of the equation $3x + 1 = x - 1$ do not have the same value for most values of x.

When $x = 2$, for example, the left member equals 7 and the right member 1. As a matter of fact, -1 is the only value of x that makes the two members of this equation equal; that is, -1 is the only number that answers the description given by this equation.

An equation that does not fit every number is called an **equation of condition**, because it imposes a condition on the numbers that satisfy it.

If each of its members is a polynomial in the letter representing the number described, the equation is called a **rational integral equation.** If neither of these polynomials is of degree greater than 1 in the letter representing the number described, the equation is said to be of the **first degree**, or **linear.**

A number that answers the description given by the equation is said to **satisfy** the equation, and is called a **root** of the equation. The process of finding the number or numbers described by an equation is called the process of solving the equation.

At the opposite extreme from identities stand descriptions (equations) that do not fit any number.

For example, $2x + 1 = 5x + 4 - 3x$.

32. Equivalent equations. — Two equations that describe the same numbers are said to be **equivalent.**

The equations $(x - 1)(x - 2) = 0$ and $(x - 1)^2(x - 2) = 0$ each describe the numbers 1 and 2; but they are not considered equivalent since the latter in a sense describes the number 1 twice. (See § 148.)

The following principles concerning equivalent equations are of frequent application in the solution of equations.

1. *If we add the same number to each member of an equation and equate the sums, the equation thus formed is equivalent to the original one.*

If we represent the left member of the equation by L and the right member by R, we have

$$L = R. \qquad (1)$$

Either L, or R, or both of them contain x.

Let the new equation be

$$L + S = R + S. \qquad (2)$$

Now any number described by equation (1), when put in place of x, makes $L = R$, and, therefore, $L + S = R + S$ (I, Chapter I). Moreover, any number described by equation (2), when put in place of x, makes $L + S = R + S$, and, therefore, $L = R$ (IV, Chapter I). That is, the two equations describe the same number and are therefore equivalent.

The case of the subtraction of the same number from each member of the equation is included in this proof since the subtraction of any number S is equivalent to the addition of $(-S)$ (3, Chapter 1).

2. *If we multiply or divide both members of an equation by the same number and equate the results, the equation thus formed is equivalent to the original one, provided that the number by which we multiply or divide is not zero and is not expressed in the terms of the unknown number of the equation.*

Let the original equation be

$$L = R, \tag{1}$$

and the new one

$$LM = RM. \tag{2}$$

Now any number described by equation (1) when put in place of x, makes $L = R$, and therefore $LM = RM$ (VII, Chapter I).

Moreover, any number described by equation (2), when put in place of x, makes $LM = RM$, and therefore $L = R$ (X, Chapter I).

That is, the two equations describe the same numbers and are therefore equivalent.

The argument just given covers the case of dividing both members by the same number, since to divide a number by M is equivalent to multiplying it by $\frac{1}{M}$.

If the M of the demonstration were 0, equation (2) would be

$$0 = 0.$$

But this is an identity and is not equivalent to (1).

The reason for saying that the number by which we multiply or divide the two members of the equation shall not be expressed in terms of the unknown of the equation will be clear from the following examples:

Consider the equation $2x + 3 = x + 2$.

If we multiply each member by $x - 3$ and equate the products, we get

$$(2x + 3)(x - 3) = (x + 2)(x - 3).$$

Now the student can readily verify that -1 is the only number that answers the first description, whereas both -1 and 3 answer the second one. Hence the two equations are not equivalent — the second one has a root which is not a root of the first one.

Consider now the equation

$$x^2 - 9 = x + 3.$$

If we divide both members by $x + 3$ and equate the quotients, we get

$$x - 3 = 1.$$

The student can verify that 4 and -3 answer the first description and that 4 is the only number answering the second one. Hence the two equations are not equivalent.

33. From Assumption I it is clear that we can transpose any term from one member of the equation to the other provided we change the sign before it.

We can find the number described by any linear equation by proceeding in the following way:

1. *If any of the numerical coefficients are fractions, multiply both members of the equation by the least common multiple of the denominators of these coefficients and equate the products.*

2. *Remove any parentheses that may occur in the resulting equation.*

3. *Transpose all the terms that involve the unknown number to the left member, and all the other terms to the right member.*

4. *Simplify each member of the resulting equation by combining like terms.*

5. *Divide both members of the last equation by the coefficient of the unknown number in the left member.*

This process gives a series of equivalent equations, and therefore the number given by the last equation must be the number described by the original equation.

It is not necessary that these steps be taken in this order.

34. Example.

Solve the equation $$\frac{7x}{3} - \frac{1}{6} = \frac{5x}{4}. \qquad (1)$$

Multiplying each member by 12,

$$28x - 2 = 15x. \qquad (2)$$

Transposing, $$28x - 15x = 2. \qquad (3)$$

Combining like terms, $$13x = 2. \qquad (4)$$

Dividing each member by the coefficient of x,

$$x = \tfrac{2}{13}.$$

This work can be checked by substituting this value of x for x in equation (1). The two members of the equation should then have the same value.

$$\frac{7 \cdot \frac{2}{13}}{3} - \frac{1}{6} = \frac{14}{39} - \frac{1}{6} = \frac{28-13}{78} = \frac{5}{26}$$

$$\frac{5 \cdot \frac{2}{13}}{4} = \frac{10}{52} = \frac{5}{26}.$$

Since the two members of this equation have the same value for this value of x, we conclude that equation (1) has been correctly solved.

EXERCISES

Solve each of the following equations or show that it is an identity, and check your results:

1. $2x + 5 = 5x + 2.$
2. $\frac{x}{2} + \frac{x}{3} = \frac{5}{4} - x.$
3. $\frac{8x-5}{10} - \frac{3}{2}(3 + 2x) = 6.$
4. $\frac{5x}{2} - 3x + 1 = \frac{2-x}{2}.$
5. $(x+1)^2 + (x+2)^2 = (x-3)^2 + (x-4)^2.$
6. $(r-1)^2 + (r-2)^2 = (r-3)^2 + (r-4)^2.$
7. $\frac{x+1}{2} + \frac{x+5}{4} + 2 = 0.$
8. $a^3 + b^3 - 3rab = 0.$ (Solve for r.)
9. $5x + 7y + 3 = 0.$ (Solve for x.)
10. $5x + 7y + 3 = 0.$ (Solve for y.)
11. $ax + by + c = 0.$ (Solve for y.)
12. $ax + by + c = 0.$ (Solve for x.)
13. $p_1 l_1 = p_2 l_2 + p_3 l_3.$ (Solve for p_2.)
14. $s - s_0 = \frac{1}{2} gt^2 - v_0 t.$ (Solve for v_0.)
15. $y^2 = 2a(x + c).$ (Solve for x.)
16. $x^2(y - px) = yp^2.$ (Solve for y.)
17. $5 - 4(x-1) + \frac{3}{4} = 3 - 4x.$
18. $p(p-y) = x(x+y).$ (Solve for y.)

19. $\frac{x}{410}+\frac{x}{330}=4+\frac{37x}{6765}$.

20. $x-\frac{x}{21\cdot 770}=\frac{27}{8\cdot 770}$.

21. $v=v_0+gt$. (Solve for t.)

22. $x(m+n-5)+3x-8=9(7+x-m)$.

23. $3(5x-4)-\frac{1}{4}(6+x)=2$.

24. $\frac{1}{5}\left(3x-\frac{1}{4}\right)+\frac{x}{3}-6\left(\frac{1}{4}-x\right)=1$.

25. $z^3-8=(z-2)(z^2+2z+4)$.

26. $y-2=m(x-1)$. (Solve for x.)

27. $mx=nW-nx$. (Solve for x.)

28. $2.5x-6.75=2.45+.75x$.

29. From each of the following equations form a new one by equating the products obtained by multiplying each member by the expression at the right. Do any of the new equations possess roots not possessed by the corresponding original equations?

(a) $x-2=5$. $x-1$.

(b) $3x+4=7x-3$. 5.

(c) $x+6=0$. x.

30. From each of the following equations form a new one by equating the quotients obtained by dividing each member by the expression at the right. Do any of the new equations possess all of the roots of the corresponding original equations?

(a) $x^2-1=0$. $x+1$.

(b) $x^2-7x=0$. x.

(c) $2x+10=0$. 2.

PROBLEMS

1. The sum of three consecutive integers is 24. What are they?

2. The sum of three consecutive integers is a. What are they? Are there any restrictions on a implied by the conditions of this problem?

3. The sum of four consecutive odd integers is a. What are they? What restrictions are imposed on a by the conditions of this problem?

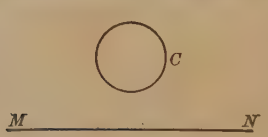
4. At what time between four and five o'clock are the hands of a watch opposite each other?

5. Can the hands of a watch be opposite each other twice within an hour?

Count the number of times the hands of a watch are opposite each other between noon and midnight.

6. How far apart must the centers of two pulleys be if each has a diameter of 16 inches, and the connecting belt is 25 feet long?

Volume of an anchor ring. — When the circle C is revolved about the line MN a solid is generated that is called an anchor ring. It is shown in Integral Calculus that the volume of this anchor ring is equal to the area of the circle multiplied by the length of the path described by the center of the circle.



7. If the radius of the circle is 3 inches, how far must the center of the circle be from the axis in order that the volume of the anchor ring shall be 1425 cubic inches?

8. What must be the outer radius of a circular ring that is 3 centimeters wide and contains 240 square centimeters?

9. What must be the outer radius of a circular ring that is 3 inches wide and contains a square inches?

What restrictions on a are implied in this problem?

10. A runs around a circular track in 30 seconds and B in 35 seconds. Two seconds after B starts, A starts from the same place in the same direction. When will they meet?

11. A runs around a circular track in a seconds, and B in b seconds. Two seconds after B starts, A starts from the same place in the same direction. When will they meet?

Describe what happens when $a = b$.

12. If in Problem 11 A starts 2 seconds before B, when will they meet?

13. If A runs around a circular track in 40 seconds, how fast must B go in order that they may meet every 18 seconds when going in opposite directions?

For the principles of the lever involved in Problems 14–18 see the Appendix.

14. A uniform beam 12 feet long weighs 14 pounds per linear foot, and has a weight of 8 pounds attached to it 5 feet from one end. When it is lying horizontal, how many pounds must a man lift in order to pick up this end of it?

15. A uniform beam 30 feet long balances on a fulcrum 1 foot 8 inches from its center when a man weighing 150 pounds stands on one end. How much does the rail weigh per linear foot?

16. The roadway of a bridge is 25 feet long, and the weight of the bridge is 5 tons. What pressure is borne by each support at the ends when a wagon weighing 2 tons is one fourth of the way across?

The bridge may be considered a lever with the fulcrum at one of the supports and the power at the other support. The measure of the power is the reaction of the support on the bridge, and is therefore equal to the pressure on this support.

If x = no. of tons pressure due to the wagon on the support nearer the wagon, then, neglecting the weight of the bridge, we have

$$25\,x = 2 \cdot \tfrac{75}{4}.$$

Hence, $$x = \tfrac{3}{2}.$$

The pressure due to the wagon on the other support is then $2 - x = \frac{1}{2}$. But the bridge itself exerts a pressure of $2\frac{1}{2}$ tons on each support. Hence the pressures on the supports are 4 tons and 3 tons respectively.

17. A uniform beam 30 feet long and weighing 175 pounds to the foot rests with its ends on two supports. What is the pressure on each of these supports when the beam carries a load of 4 tons 6 feet from one end?

18. A beam is supported at its ends and a weight of W pounds is placed on it m feet from one end and n feet from the other. Find the pressure on each of the supports due to the weights.

19. A freight train whose rate is 18 miles per hour passes a certain station h hours and m minutes before a passenger train. What must be the rate of the passenger in order that it may overtake the freight in t hours?

CHAPTER V

SYSTEMS OF LINEAR EQUATIONS IN TWO OR MORE UNKNOWNS

35. We have seen that a conditional equation of the first degree in one unknown is a description of a number and that there is only one number that answers such a description. In this chapter we shall see that an equation of the first degree in two unknowns is a description of a relation between two numbers, but that this description differs from the one given by an equation in one unknown in that there are a great many numbers related to each other in the way described by the equation.

Consider, for example, the equation

$$5x + 2y = 4.$$

The student can readily verify that any pair of values of x and y in the following table are related to each other in the way described by this equation:

x	-8	0	2	$\frac{7}{2}$	-3	$-\frac{2}{3}$	1	6	—	—	—	—	—	—
y	22	2	-3	$-\frac{27}{4}$	—	—	—	—	0	-7	-1	-1	5	6

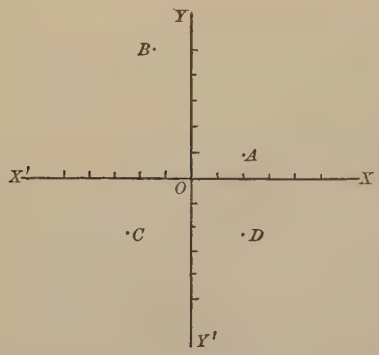
The student should complete the table by finding the values of y that are related to the second four values of x in the given way, and finding the values of x that are related to the last six values of y in the given way.

36. Graphical representation. — A description of the relation between two numbers by means of an equation can also be given by means of a graphical representation.

The following explanations and illustrations are given with a view to enabling the student to become familiar with this method of representing these relations.

In Chapter I we looked upon the real numbers as the representatives of the points of a straight line with reference to a

given origin and a given unit length. In a similar way pairs of real numbers can be looked upon as the representatives of the points of a plane. We take two perpendicular straight lines in this plane and their point of intersection as the origin on each line. If x and y are a pair of numbers we lay off x on the horizontal line, in the way just recalled, and y in a similar way on the vertical line, counting distances above the origin as positive and those below as negative. Then through the point representing each of these numbers we draw a line perpendicular to the line on which the point lies. The point of intersection of these two perpendiculars is the point represented by the pair of numbers x and y.



In the figure given above, the points A, B, C, and D are represented by the pairs of numbers $(2, 1)$; $(-\frac{3}{2}, 5)$; $(-\frac{5}{2}, -\frac{7}{3})$; and $(2, -\frac{7}{3})$, respectively. We can also say that these four points represent these four pairs of numbers respectively.

37. Definitions. — If the numbers x and y represent the point A in the way just described, x is called the **abscissa** of A, and y its **ordinate**. The two numbers together are called the **coördinates** of the point.

The abscissa of a point is always given first. Thus, the point $(2, 3)$ is the point whose abscissa is 2 and ordinate 3.

The two perpendicular lines are called the **axes of coördinates**, or **coördinate axes.**

The axis on which the abscissæ are laid off is called the ***x*-axis**, and the one on which the ordinates are laid off is called the ***y*-axis.** The x-axis is usually horizontal.

38. Every pair of values of x and y is represented by a point, and every point represents a pair of values of x and y, which are its coördinates. These can be found by drawing perpendiculars from the point to the axes and measuring the distances from the origin to the feet of these perpendiculars.

EXERCISES

Draw two axes and locate the following points, using $\frac{1}{2}$ inch or 1 centimeter as the unit distance:

1. $(2, 3)$, $(5, -2)$, $(0, 1)$.

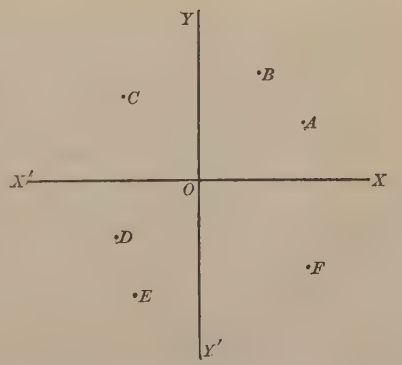
2. $(-2, -2)$, $(-3, -3)$, $(-\frac{5}{2}, -\frac{5}{2})$, $(1, 1)$, $(4, 4)$.

3. $(-\frac{3}{2}, \frac{3}{2})$, $(0, 0)$, $(3, -3)$, $(\frac{5}{2}, -\frac{5}{2})$.

4. $(7, 0)$, $(5, 0)$, $(4, 0)$, $(1, 0)$, $(-3, 0)$, $(-8, 0)$.

5. $(0, -10)$, $(0, -8)$, $(0, -7)$, $(0, -4)$, $(0, 1)$, $(0, 2)$, $(0, 3)$.

6. Determine by accurate measurements the coördinates of the points A, B, C, D, E, and F in the accompanying figure.

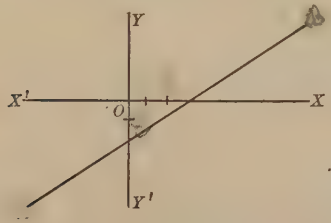


7. Draw a line parallel to the x-axis and 3 units above it, and measure the coördinates of 5 points on it.

8. Draw a line parallel to the y-axis and 2 units to the left of it, and measure the coördinates of 5 points on it.

9. Lay off the points whose coördinates are the successive pairs of values of x and y in the table of § 35. Can a straight line be drawn through these points?

39. Locus of an equation. — For a given equation in x and y there is a certain line, straight or curved, such that the coördinates of every point on the line satisfy the equation, and that every pair of values of x and y that satisfy the equation are the coördinates of a point on the line.



This line is called the **locus** of the equation. It exhibits graphically the relation between the unknowns (or variables) that is described by the equation in the sense that those numbers which are the coördinates of points of this line, and only those numbers, are related to each other in the way described by the equation.

For example, the locus of the equation

$$2x - 3y = 6$$

is the straight line of the accompanying figure.

EXERCISES

Lay off carefully 10 points on the locus of each of the following equations:

1. $x = y$.
2. $x - y = 5$.
3. $2x = 3y$.
4. $2x + 7y + 4 = 0$.
5. $-5x + 6y + 7 = 0$.
6. $x + y + 1 = 0$.
7. $x = 0$.
8. $y = 0$.
9. $x + y = 0$.
10. $y = 3x$.
11. $\frac{x}{2} + \frac{y}{3} = 1$.
12. $-\frac{x}{4} + \frac{y}{6} = 1$.
13. $\frac{x}{3} - y = 1$.
14. $-\frac{x}{5} - \frac{y}{8} = 1$.
15. $4x + 5 = 0$.
16. $x - 4 = 0$.
17. $y + 5 = 0$.
18. $3x + 5y = 2$.
19. $y - 1 = 3(x - 2)$.
20. $y + 3 = -5(x - 1)$.

40. Graphical solution. — It is proved in Analytic Geometry that *the locus of every equation of the first degree in two variables is a straight line.*

It is because of this fact that equations of the first degree are called *linear* equations.

Any pair of values of the unknowns that satisfy each of two linear equations in two unknowns must be the coördinates of a point on the locus of each equation; that is, must be the coördinates of the point of intersection of these loci. Hence, since two straight lines intersect in not more than one point, two linear equations in two unknowns cannot be satisfied by more than one pair of values of the unknowns unless they are dependent. (See § 42.) Thus we see that although a single linear equation in two unknowns does not describe two particular numbers, two such equations taken together do give us such a description, provided that they are not dependent and not inconsistent. (See § 42.)

The values of the unknowns that satisfy two linear equations can be found by plotting the loci of these equations and measuring the coördinates of their point of intersection; but results obtained in this way are only approximate since the locus of an equation cannot be drawn with perfect accuracy and since the coördinates of a point of intersection of two loci cannot be measured with perfect accuracy.

41. Algebraic solution. — These values of the unknowns can also be found without reference to the loci of the equations.

We shall illustrate two ways of doing this:

1. Solution by the method of substitution.

Find the numbers described by the equations

$$3x - 4y = 7, \tag{1}$$

$$7x + 5y = 9. \tag{2}$$

Solve (1) for x,

$$x = \frac{7 + 4y}{3}. \tag{3}$$

Substitute this value of x for x in (2),

$$\frac{7(7+4y)}{3}+5y=9. \tag{4}$$

Solve (4) for y, $49+28y+15y=27,$

$$43y=-22,$$

$$y=-\frac{22}{43}.$$

Substitute this value of y, for y in (3),

$$x=\frac{7-\frac{88}{43}}{3}=\frac{71}{43}.$$

Check. — From (1),

$$3\cdot\frac{71}{43}-4\left(-\frac{22}{43}\right)=\frac{301}{43}=7.$$

From (2), $\quad 7\cdot\frac{71}{43}+5\left(-\frac{22}{43}\right)=\frac{387}{43}=9.$

From a study of this example, the student should formulate a general rule of procedure.

2. Solution by the method of addition or subtraction.

Multiply each member of equation (1) by 5 and each member of (2) by 4.

$$15x-20y=35. \tag{5}$$

$$28x+20y=36. \tag{6}$$

The sum of the left members of (5) and (6) equals the sum of the right members. Hence,

$$43x=71,$$

$$x=\frac{71}{43}.$$

Substitute this value of x for x in (1),

$$\frac{213}{45}-4y=7,$$

$$213-172y=301,$$

$$172y=-88,$$

$$y=-\tfrac{22}{43}.$$

The check is the same as in the other method.

From a study of this example the student should formulate a general rule of procedure.

42. Inconsistent and dependent equations. — Sometimes two linear equations in two unknowns have no common solution. This is the case when the loci of the two equations are parallel lines. The student should verify that

$$3x + 4y = 1,$$

and

$$3x + 4y = 5$$

are two such equations.

Such equations are said to be **inconsistent.**

Sometimes two equations convey the same information concerning the relations of the unknowns. Such equations are said to be **dependent.** Two dependent equations have the same locus.

The equations

$$3x + 4y = 1,$$

and

$$6x + 8y = 2$$

are dependent equations. The second one gives no information concerning x and y that is not given by the first one.

EXERCISES

Solve the following pairs of equations graphically and also by one of the other methods. See that the results agree approximately.

1. $2x - y = 1,$
   $3x + 2y = 12.$

2. $x + y + 3 = 0,$
   $4x - 5y + 3 = 0.$

3. $5x + 2y = 2,$
   $7x - 5y = 34.$

4. $2x + 3y = 6,$
   $\frac{x}{2} + \frac{y}{4} = 1.$

5. $\frac{x}{3} - \frac{y}{5} = -1,$
   $-\frac{x}{4} + \frac{y}{2} = 1.$

6. $3x + 4y + 5 = 0,$
   $6x + 8y = -10.$

7. $2x - 3y + 4 = 0,$
   $3y - 2x - 2 = 0.$

8. $3 - 4x = y,$
   $8x + 2y = 6.$

9. $\dfrac{y+2}{3} = x - 2,$
$y - 4 = \dfrac{x-5}{-8}.$

10. $\dfrac{x+4}{6} = \dfrac{y-5}{-9},$
$3x - 2y + 2 = 0.$

11. $x + y + 1 = 0,$
$2x + 3y + 4 = 0.$

12. $3x + 4y - 1 = 0,$
$12x - 5y + 6 = 0.$

13. $5x - 2y + 11 = 0,$
$6 - \frac{1}{2}x = \dfrac{y-5}{4}.$

14. $\dfrac{x}{a} + \dfrac{y}{b} = 1,$
$ax + by = c.$

The lines of Ex. 14 cannot be plotted until definite values are assigned to a, b, and c.

15. $ax + by + c = 0,$
$lx + my + n = 0.$

16. $\dfrac{x}{a} + \dfrac{y}{b} = 1,$
$\dfrac{x}{a} - \dfrac{y}{b} = 1.$

17. $\dfrac{x}{m} + \dfrac{y}{n} = 1,$
$\dfrac{x}{4m} - \dfrac{y}{5n} = 1.$

In solving the following equations do not clear of fractions, but solve for $\dfrac{1}{x}$ and $\dfrac{1}{y}$. These equations are not linear and their loci are not straight lines. Do not try to solve them graphically.

18. $\dfrac{2}{x} + \dfrac{5}{y} = 3,$
$\dfrac{1}{x} - \dfrac{4}{y} = 2.$

19. $-\dfrac{7}{x} + \dfrac{9}{y} = 2,$
$\dfrac{12}{x} + \dfrac{4}{y} = 1.$

20. $\dfrac{a}{x} + \dfrac{b}{y} = c,$
$\dfrac{d}{x} + \dfrac{e}{y} = f.$

43. Solution by formulæ. — Consider the two equations

$$a_1x + b_1y = c_1, \tag{1}$$
$$a_2x + b_2y = c_2. \tag{2}$$

The student should verify that if $a_1b_2 - a_2b_1 \neq 0$ the numbers described by these equations are

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},$$

and

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

These formulæ can be easily remembered by means of the following device:

The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ will be used to represent the difference of the cross products a_1b_2 and a_2b_1. That is, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

For example,

$$\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix} = 6 - 20 = -14, \text{ and } \begin{vmatrix} \frac{2}{3} & -8 \\ 7 & \frac{1}{2} \end{vmatrix} = \frac{1}{3} - (-56) = 56\tfrac{1}{3}.$$

This symbol is called a **determinant of the second order.**

The values of x and y which were obtained as the solution of the preceding equations can now be expressed as follows:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

44. We observe that the two denominators are the same determinant. The numbers in the first row of this determinant are the coefficients of x and y respectively in the first equation, and the numbers in the second row are the coefficients of x and y in the second equation. The numerator of the fraction that gives the value of x is the same as the denominator with the exception that the coefficients of x (the numbers in the first column) have been replaced by the constant terms of the respective equations; and in the fraction that gives the value of y the numerator is the same as the denominator with the exception that the coefficients of y (the numbers in the second column) have been replaced by the constant terms of the respective equations.

By means of these formulæ we can write down immediately the solutions of any pair of consistent, independent, linear equations in two unknowns.

For example, the values of x and y that satisfy the equations

$$7x + 12y = 3$$

and

$$x - 2y = 5$$

are

$$x = \frac{\begin{vmatrix} 3 & 12 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 7 & 12 \\ 1 & -2 \end{vmatrix}} = \frac{-6 - 60}{-14 - 12} = \frac{33}{13},$$

and

$$y = \frac{\begin{vmatrix} 7 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 7 & 12 \\ 1 & -2 \end{vmatrix}} = \frac{35 - 3}{-14 - 12} = -\frac{16}{13}.$$

The student should observe that these formulæ apply only when each equation is in the form $ax + by = c$. If it is not in this form originally, it must first be reduced to this form. If one of the letters does not appear in an equation, it should be regarded as present with the coefficient zero. (See Ex. 11.)

In solving Equations (1) and (2) of § 43 we get

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

and

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c.$$

The common coefficient of x and y in these equations is the determinant in the denominators of the formulæ, while the right members of the equations are the determinants in the numerators of the formulæ.

Hence if in these formulæ the determinant in the denominators is zero, while those in the numerators are different from zero, the equations are inconsistent. If all the determinants are zero, the equations are dependent.

EXERCISES

Solve the following pairs of equations by means of the formulæ:

1. $9x + y = 5$,
   $3x + 17y = 11$.

2. $\frac{x}{5} + \frac{y}{7} = 1$,
   $\frac{x}{3} - \frac{y}{4} = 1$.

3. $y - 5 = 4x$,
   $y - 3 = 7(x + 1)$.

4. $y - \frac{2}{3} = \frac{7}{2}(x + \frac{1}{4})$,
   $y - \frac{1}{5} = \frac{4}{9}(x - \frac{2}{3})$.

5. $\frac{y - 7}{5} = \frac{x + 10}{3}$,
   $\frac{x - 4}{2} = \frac{y - 8}{3}$.

6. $4x + 3y = 2$,
   $8x + 6y = 1$.

7. $-\frac{x}{5} + \frac{y}{2} = 1$,
   $\frac{x}{3} + \frac{y}{7} = 1$.

8. $\frac{x}{4} + \frac{y}{2} = 1$,
   $x + 2y = 4$.

9. $\frac{x - 1}{3} = \frac{y - 2}{-8}$,
   $x + \frac{y}{2} = 1$.

10. $3x - 5y = 2$,
    $1\frac{1}{2}x = 2\frac{1}{2}y + 1$.

11. $5x + 19y = 14$,
    $3x = 2$.

12. $kx + ly + m = 0$,
    $2x - 3y = 4m$.

13. $a(x - y) + b(x + y) = b + c$,
    $a(x + y) - b(x - y) = b - c$.

14. $\frac{x}{a} + \frac{y}{b} = 1$,
    $\frac{x}{2a} + \frac{y}{4b} = 1$.

15. $(a + b)x + (a - b)y = 3ab$,
    $(a - b)x + (a + b)y = ab$.

16. $ax + by + c = 0$,
    $ax - by - c = 0$.

17. $lx + my = n$,
    $2lx + 3my = n$.

18. $3x + 1 = 5$,
    $x + y = 2$.

19. $a(x + y) + b(x - y) = a$,
    $(a + b)x + (a - b)y = b$.

20. $\frac{x}{a} + \frac{y}{b} = 1$,
    $\frac{x}{c} + \frac{y}{d} = 1$.

45. Two descriptions of the relations connecting three unknown numbers are not sufficient to identify these numbers.

For example, there are many sets of values of x, y, and z that answer the two following descriptions:

$$2x + 3y + z = 4, \tag{1}$$

$$-4x + 5y + 6z = 7. \tag{2}$$

The student should make a table exhibiting at least five sets of values of x, y, and z that answer these descriptions. How many more such sets of values are there?

There is however only one set of values of the unknowns that answer three descriptions of the first degree of the relations connecting them. Consider, for example, the description

$$x + y + z = 3, \tag{3}$$

in connection with (1) and (2).

If we multiply each member of (1) by 2 and add each member of the resulting equation to the corresponding member of (2), we get

$$11y + 8z = 15. \tag{4}$$

Then if we multiply each member of (3) by 2 and subtract the products from (1), we get

$$y - z = -2. \tag{5}$$

Now every set of values of x, y, and z that satisfy equations (1), (2), and (3) must also satisfy equations (4) and (5). But the only values of y and z that satisfy (4) and (5) are

$$y = -\tfrac{1}{19},$$

$$z = \tfrac{37}{19}.$$

Moreover if y and z have these values, we must have $x = \frac{21}{19}$ in order that (1), (2), and (3) be satisfied. Hence there is only one set of values of x, y, and z that satisfy equations (1), (2), and (3).

It was not necessary to proceed in exactly this way in order to get the solution of the given equations. We might have

chosen a different unknown for the first elimination and accordingly have combined the original equations differently. But the same variable must always be eliminated in the first two eliminations.

The work of solving a system of three linear equations in three unknowns can be checked, as in the case of equations with two unknowns, by substituting in the equations the values found for the unknowns.

46. Inconsistent and dependent equations. — It may be that there are no values of x, y, and z that satisfy all three equations.

Consider, for example, the following equations:

$$2x + 3y + z = 4, \quad (1)$$
$$x + y + z = 3, \quad (2)$$
$$4x + 6y + 2z = 5. \quad (3)$$

If we multiply each member of (1) by 2 and subtract the products from the corresponding members of (3), we get

$$0 = -3.$$

This obvious untruth was obtained on the supposition that there is a set of values of x, y, and z that satisfy both (1) and (3). Hence the supposition must be wrong and (1) and (3) are inconsistent.

Exercise. — Explain in detail where the existence of a set of values of x, y, and z satisfying (1) and (3) was assumed in getting the result $0 = -3$.

Sometimes the three equations are such that one of them gives no more information concerning the relations connecting the unknowns than is given by the other two equations.

The equations

$$2x + 3y + 4z = 5, \quad (1)$$
$$-x + 2y - 3z = 2, \quad (2)$$
$$x + 5y + z = 7, \quad (3)$$

form such a set, inasmuch as the two members of (3) are the sums of the corresponding members of (1) and (2), and any set of values of x, y, and z satisfying (1) and (2) must also satisfy (3).

The student should give to z in (1) and (2) any convenient value and solve the resulting equations for x and y. He will then find that these values of x, y, and z satisfy (3).

Such equations are said to be **dependent**. If each of the equations contains information not contained in the others, the equations are said to be **independent.**

NOTE. — If the student has studied solid geometry, he may be interested in the fact that an equation of the first degree in three variables represents a plane, just as an equation of the first degree in two variables represents a straight line. The loci of two inconsistent equations are parallel planes. The loci of three dependent equations are three planes through the same line.

EXERCISES

Solve the following sets of equations and check your results:

1. $x+y+z=6,$
   $2x+3y-z=5,$
   $4x+2y+2z=14.$

2. $2x-4y-6z=-6,$
   $3x+5y+2z=13,$
   $-x+2y+z=0.$

3. $7x+5y+8z=3,$
   $-6x+2y+5z=6,$
   $14x+10y+16z=4.$

4. $x+y+1=0,$
   $y+z=4,$
   $z+x=1.$

5. $-x-y+2z=12,$
   $4x-3y-2z=10,$
   $2x+4y+3z=3.$

6. $ax+by=1,$
   $cy+dz=1,$
   $ez+fx=1.$

7. $4A-5B+6C=23,$
   $2A+7B+3C=40,$
   $9A+B+8C=61.$

8. $\frac{A}{3}+\frac{B}{2}+\frac{C}{5}=2,$
   $\frac{A}{2}+\frac{B}{5}+\frac{C}{3}=4,$
   $\frac{A}{5}+\frac{B}{3}+\frac{C}{2}=\frac{63}{5}.$

9. $\frac{1}{x}+\frac{1}{y}=l,$
   $\frac{1}{y}+\frac{1}{z}=m,$
   $\frac{1}{z}+\frac{1}{x}=n.$

   Solve for $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$.

10. $A+B+C=3,$
    $A+B-C=2,$
    $A-B+C=5.$

11. $A+B=0,$
    $B+C=1,$
    $C+A=3.$

12. $14g+20f+c=-149,$
$16g+18f+c=-145,$
$6g+4f-c=13.$

13. $c=0,$
$2f+c+1=0,$
$2g+c+1=0.$

14. $5+2g+4f+c=0,$
$5+4g+2f+c=0,$
$13+6g-4f+c=0.$

15. $x+y+z=3,$
$x-2y+z=0,$
$3x+4y-z=6.$

16. $\frac{x}{2}+\frac{y}{3}+z=1,$
$x-\frac{y}{3}-\frac{z}{2}=1,$
$-\frac{x}{3}-y+\frac{z}{2}=1.$

17. $3A+2B+4C+4=0,$
$2A-5B-C=20,$
$A+B-C=0.$

18. $4x-3y-4z=0,$
$5x+6y+10z=7,$
$x+2z=1.$

19. $12x+5y+8z=-14,$
$7x+y+z+\frac{5}{2}=0,$
$x+y+2z=-3.$

20. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3,$
$\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=1,$
$-\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=-1.$

21. $a_1x+b_1y+c_1z=d_1,$
$a_2x+b_2y+c_2z=d_2,$
$a_3x+b_3y+c_3z=d_3.$

47. Solution by formulæ. — The equations in Ex. 21 of the preceding article are typical of every set of three simultaneous linear equations in three unknowns, since any such set can be obtained from these by assigning proper values to a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, a_3, b_3, c_3, and d_3. Hence the values of x, y, and z that satisfy these equations constitute formulæ for the solution of any set of three linear equations in three unknowns.

If $a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3\neq 0$ these values of x, y, and z are,

$$x=\frac{d_1b_2c_3+d_2b_3c_1+d_3b_1c_2-d_3b_2c_1-d_1b_3c_2-d_2b_1c_3}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3},$$

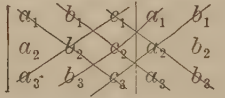
$$y=\frac{a_1d_2c_3+a_2d_3c_1+a_3d_1c_2-a_3d_2c_1-a_1d_3c_2-a_2d_1c_3}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3},$$

$$z=\frac{a_1b_2d_3+a_2b_3d_1+a_3b_1d_2-a_3b_2d_1-a_1b_3d_2-a_2b_1d_3}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3}.$$

These formulæ can be remembered easily by means of the following device:

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ will be used to represent the expression $a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3$.

The six terms of this expression are formed in the following way: the first and second columns of the symbol just described are rewritten at the right of the symbol thus:



Then the products of the numbers in the diagonals running down from left to right are formed, and also of the numbers in the diagonals running down from right to left. The signs of these last three products are changed. The algebraic sum of these six products with their signs thus modified is the value of the symbol.

For example,

$$\begin{vmatrix} 2 & 3 & 1 \\ 5 & 2 & -6 \\ 1 & 4 & 7 \end{vmatrix} = 2\cdot 2\cdot 7 + 3(-6)\cdot 1 + 1\cdot 5\cdot 4 - 1\cdot 2\cdot 1 - 2(-6)\cdot 4 - 3\cdot 5\cdot 7$$
$$= 28 - 18 + 20 - 2 + 48 - 105$$
$$= -29.$$

This symbol is called a **determinant of the third order.**

The values of x, y, and z that were obtained as the solution of the given equations can now be expressed as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

48. We observe that the three denominators are the same determinant. The numbers in the first row of this determinant

are the coefficients of x, y, and z respectively in the first equation, and the numbers in the second and third rows are the respective coefficients of these unknowns in the second and third equations respectively. The numerator of the fraction that gives the value of x is the same as the denominator with the exception that the coefficients of x (the numbers in the first column) have been replaced by the constant terms of the respective equations. The numerators of the fractions giving the values of y and z are related to their respective denominators in an analogous way.

49. Suppose we wish to solve the equations

$$x+y+z-1=0,$$
$$x-y-z=1,$$
$$2x+3y=8,$$

by means of these formulæ.

We observe that the first equation is not in the form assumed in the preceding discussion — the constant term must be in the right member. Hence we rewrite the equations as follows:

$$x+y+z=1,$$
$$x-y-z=1,$$
$$2x+3y=8.$$

Then, by the formulæ,

$$x=\frac{\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 8 & 3 & 0\end{vmatrix}}{\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 2 & 3 & 0\end{vmatrix}}=\frac{3-8+8+3}{3-2+2+3}=1,$$

$$y=\frac{\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 2 & 8 & 0\end{vmatrix}}{\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 2 & 3 & 0\end{vmatrix}}=\frac{8-2-2+8}{6}=2,$$

$$z=\frac{\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 2 & 3 & 8\end{vmatrix}}{\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 2 & 3 & 0\end{vmatrix}}=\frac{-8+3+2+2-3-8}{6}=-2.$$

50. It can be proved that if the determinant in the denominators in the formulæ at the end of § 47 is zero and the determinants in the numerators are not all zero, the equations are inconsistent.

Consider, for example, the equations

$$\begin{aligned} x + y - 2z &= 1, \\ 3x + 4y + 5z &= 4, \\ 5x + 6y + z &= 9. \end{aligned}$$

It can be shown that the equations may also be inconsistent when all the determinants of these formulæ are zero. The situation here differs in this respect from what we found in the case of two linear equations in two unknowns. (§ 44.)

Consider, for example, the equations

$$\begin{aligned} 3x + 4y - 2z &= 1, \\ 6x + 8y - 4z &= 4, \\ 9x + 12y - 6z &= 5. \end{aligned}$$

EXERCISES

Solve the following sets of equations by the formulæ:

1. $x+2y+z=0,$
   $x-2y-8=0,$
   $x+y+z=3.$

2. $x+y+2z=0,$
   $2x-y-2z-1=0,$
   $3x+6z-5y+1=0.$

3. $A+B+C=3,$
   $5A+4B+3C=5,$
   $6A+3B+2C=2.$

4. $A+C=1,$
   $B+A=3,$
   $2C-B=1.$

5. $2x-3y+4z=1,$
   $x+y-5z=2,$
   $3x-2y-z=3.$

6. $2l-3n+m=2,$
   $l+m+n+1=0,$
   $l-4m-4n=5.$

7. $-4A-5B+C=7,$
   $2A+3C=10,$
   $-5B+7C=3.$

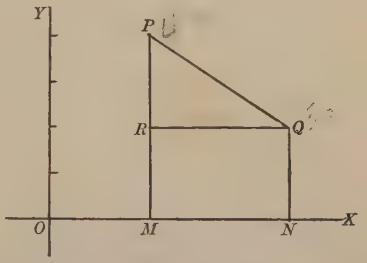
8. $A+B=2$,
$B+C=4$,
$C+A=3$.

9. $x-y+z=2$,
$y-z=-3$,
$z-x=6$.

10. $\frac{x}{2}+\frac{y}{4}-\frac{z}{5}=3$,
$x+\frac{y}{6}+\frac{z}{2}=11$,
$\frac{x}{4}+\frac{y}{3}+\frac{z}{10}=6$.

51. Geometrical applications. — The discussion of geometrical principles given in this article and the following one leads up to an important geometrical use of systems of three linear equations in three unknowns.

Denote the two points (2, 4) and (5, 2) by P and Q respectively. Draw PM and QN parallel to the y-axis and QR parallel to the x-axis. Connect P and Q.



Now, $RQ=MN=5-2$, since $OM=2$ and $ON=5$; and $RP=4-2$, since $MP=4$ and $MR=2$. Hence,

$$PQ=\sqrt{(5-2)^2+(4-2)^2}=\sqrt{13}.$$

In general, if P represents the point (x_1, y_1) and Q the point (x_2, y_2), $RQ=x_2-x_1$ and $RP=y_1-y_2$. Hence,

$$PQ=\sqrt{(x_2-x_1)^2+(y_1-y_2)^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

If we represent the distance PQ by d, we have the formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

This important formula enables us to find the distance between two points when the coördinates of these points are given.

EXERCISES

Find the distances between the following pairs of points:

1. $(1, 3), (2, 5)$.
2. $(0, 0), (4, -3)$.
3. $(-2, -7), (-3, 6)$.
4. $(10, 0), (0, 2)$.
5. $(8, -3), (-3, 8)$.
6. Show that the quadrilateral whose vertices are the points $(3, 2), (-4, 2), (-4, -5)$, and $(3, -5)$ respectively has equal sides.

52. The equation of a circle. — Every point in the circumference of a circle whose radius is 5 and whose center is at the point $(2, 3)$ is 5 units distant from the center.

Hence if x and y are the coördinates of such a point, we have

$$(x-2)^2+(y-3)^2=25.$$

If x and y are the coördinates of a point inside the circle, then

$$(x-2)^2+(y-3)^2<25;$$

and if they are the coördinates of a point outside the circle, then

$$(x-2)^2+(y-3)^2>25.$$

The points in the circumference of this circle are therefore the only points whose coördinates answer the description given by this equation. We say accordingly that this equation is the equation of the circle and that the circle is the locus of this equation.

EXERCISES

Write the equations of the following circles, and draw the circles:

1. Radius 3 and center at $(-2, 1)$.
2. Radius 1 and center at $(0, 0)$.
3. Radius 4 and center at $(4, 0)$.
4. Radius 6 and center at $(0, 6)$.

5. Radius 2 and center at (2, 2).
6. Radius 5 and center at $(-3, -4)$.
7. Radius 3 and center at $(5, -2)$.
8. Radius 4 and center at (0, 0).
9. Radius 7 and center at (5, 3).
10. Radius $3\frac{1}{2}$ and center at $(7, -6)$.

53. The equation $(x-h)^2+(y-k)^2=r^2$,

which is the equation of the circle with the radius r and the center at the point (h, k), can be written in the form

$$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0.$$

This is equivalent to the form

$$x^2+y^2+2gx+2fy+c=0,$$

if we let $2g$ stand for $-2h$, $2f$ stand for $-2k$, and c stand for $h^2+k^2-r^2$.

The equation of every circle can be written in this form.

We can take advantage of this fact to find the equation of the circle that passes through three given points not on the same straight line.

For example, suppose that we want to find the equation of the circle that passes through the points (7, 10), (8, 9), and $(-3, -2)$.

The equation must be in the form

$$x^2+y^2+2gx+2fy+c=0,$$

and it must be satisfied by the coördinates of each of these points. Hence,

$$\begin{aligned} 49+100+14g+20f+c&=0,\\ 64+81+16g+18f+c&=0,\\ 9+4-6g-4f+c&=0. \end{aligned}$$

Or

$$\begin{aligned} 14g+20f+c&=-149,\\ 16g+18f+c&=-145,\\ 6g+4f-c&=13. \end{aligned}$$

This is a system of three linear equations in the three unknowns, g, f, and c. If we solve these equations by either of the methods already considered, we get

$$g = -2,\ f = -4,\ c = -41.$$

Hence the equation of the circle sought is

$$x^2 + y^2 - 4x - 8y - 41 = 0.$$

The student should verify directly that this equation is satisfied by the coördinates of the given points.

PROBLEMS

1. On the Centigrade thermometer the temperature of melting ice is marked 0° and that of boiling water 100°. Write down the equation giving the relation between the reading of the Centigrade and the Fahrenheit thermometers, and give a graphical means of changing from one of these readings to the other by drawing the locus of this equation.

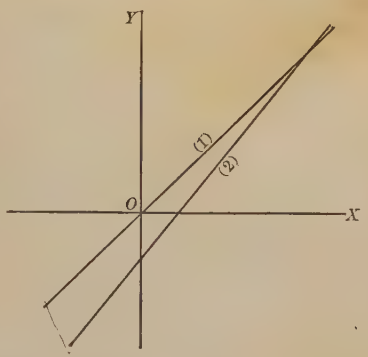
2. The following temperatures (Fahrenheit) were reported by the New York *Sun* for Jan. 16, 1912.

	Degrees		Degrees
Midnight	13	8 A.M.	3
1 A.M.	10	9 A.M.	2
2 A.M.	8	10 A.M.	6
3 A.M.	8	11 A.M.	10
4 A.M.	6	Noon	12
5 A.M.	4	1 P.M.	11
6 A.M.	4	2 P.M.	13
7 A.M.	3	3 P.M.	13
7.30 A.M.	2	4 P.M.	13

From the figure of the preceding problem measure off the corresponding temperatures on the Centigrade thermometer.

3. A train passes a station at the rate of 10 miles an hour. Half an hour afterwards another train follows it at the rate of 12 miles an hour. Determine graphically how long it will take the latter to overtake the former.

Let y = number of miles traveled by the first train in x hours. Then $y = 10x$. The locus of this equation is the line marked (1) in the figure.



It represents graphically the relation between the distance traveled and the time of traveling of the first train. When the first train has traveled x hours, the second one has traveled $x - \frac{1}{2}$ hours. If now we let $y =$ no. of miles traveled by the second train, $y = 12\,(x - \frac{1}{2})$. The locus of this equation is the line marked (2) in the figure. The abscissa of the point of intersection of these two loci represents the number of hours from the time of departure of the first train to the time of meeting.

4. Determine graphically how long it will take an automobile going 25 miles an hour to overtake another one that had 15 miles start and is going at the rate of 16 miles an hour.

5. Determine graphically where two trains will meet if one of them starts from Detroit for Chicago at the rate of 30 miles an hour at the same time the other one starts from Chicago for Detroit at the rate of 40 miles an hour. The distance from Detroit to Chicago is 285 miles.

6. A man rows 9 miles downstream in 2 hours and returns in 4 hours. How fast does he row in still water, and what is the rate of the current?

In solving this problem assume that the rate downstream is equal to the rate of rowing in still water increased by the rate of the stream, and that the rate upstream is equal to the difference of these two rates.

7. A and B working together do one fifth of a piece of work in a day. After they have worked together three days, B stops and A finishes in three days more. How long would it have taken each one to do the work alone?

8. A glass of water weighs 14 ounces. When full of sulphuric acid of specific gravity 1.35, the glass weighs

17.5 ounces. Find the weight of the glass when empty and the number of ounces of water it holds.

See the Appendix for an explanation of specific gravity.

9. A certain vessel weighs p pounds when full of a substance of specific gravity s_1, and it weighs q pounds when full of a substance of specific gravity s_2. What is the weight of the vessel, and how much of each substance does it hold?

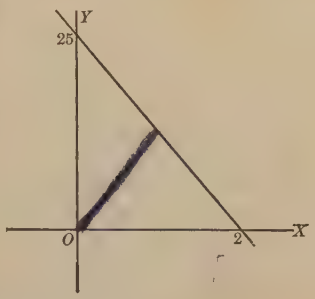
10. Two weights of 25 pounds and 30 pounds respectively balance when resting on a lever at unknown distances from the fulcrum. If 5 pounds are added to the first weight, the other one must be moved 2 feet further from the fulcrum to maintain the balance. What was the original distance from the fulcrum to each of the weights?

11. Two weights of w_1 and w_2 pounds respectively balance when resting on a lever at unknown distances from the fulcrum. If p pounds are added to the first weight, the other one must be moved f feet further from the fulcrum to maintain the balance. What was the original distance from the fulcrum to each of the weights?

12. A weight of 20 pounds is placed at random on a beam of unknown length that is supported at its ends. It is found that this produces a pressure of 15 pounds on the support at one end in addition to that caused by the weight of the beam. It is also found that the same pressure is produced on this support by a weight of 25 pounds placed 2 feet further from this end. How long is the beam?

13. A weight of p pounds is placed at random on a beam of unknown length that is supported at its ends. It is found that this produces a pressure of a pounds on the support at one end in addition to that caused by the weight of the beam. It is also found that the same pressure is produced on this support by a weight of $p+n$ pounds placed m feet further from this end. How long is the beam?

14. Represent graphically the change in pressure borne by each support of the bridge described in Problem 16, p. 45, as the wagon moves across the bridge.



Let $x =$ the number of tons pressure due to the wagon on the first support.

Let $y =$ the number of feet from the first support to the wagon.

Then $25\,x = 2(25 - y)$,

or $25\,x + 2\,y = 50$.

The point on the locus of this equation whose ordinate is the distance of the wagon at any moment from the first support has for abscissa the pressure in tons on this support at this moment.

15. A beam 18 feet long is supported at its ends. Represent graphically the change in the pressure on each support as a weight of 150 pounds moves across the beam.

16. A locomotive weighing 600 tons moves across a bridge 70 feet long that is supported by stone abutments at the ends. Represent graphically the change in the pressure on the two supports.

Find the equation of the circle that passes through each of the following sets of three points. Draw each circle.

17. (3, 2), (1, 4), (− 2, 1).

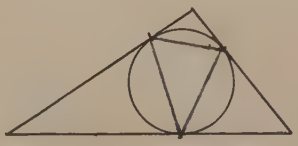
18. (0, 0), (3, 0), (0, 5).

19. (− 2, 1), (3, 4), (4, − 2).

20. (− 5, − 5), (3, 3), (4, 0).

21. (− 3, − 6), (2, 7), (4, 8).

22. Find the distances from the vertices to the points of contact of the circle inscribed in the triangle whose sides are 9, 14, and 18 respectively.

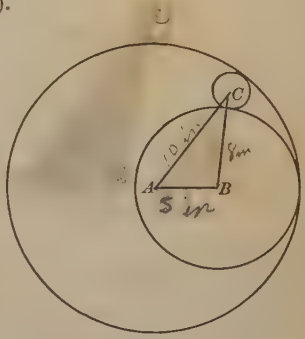


23. Find the distances from the vertices to the points of contact of the circle inscribed in the triangle whose vertices are $(-21, -22)$, $(11, 2)$, $(-14, 2)$.

24. Three circles are drawn with centers at the points $(-26, -39)$, $(46, -9)$, and $(-10, 24)$ respectively and such that they are tangent to one another externally. What are the equations of these circles?

25. Find the equations of the three circles that are tangent to one another externally and have their centers at the points $(-16, -6\frac{2}{3})$, $(5, 2\frac{1}{12})$, and $(-5, 12)$.

26. In the triangle ABC, $AB = 5$ inches, $BC = 8$ inches, and $CA = 10$ inches. What must be the radii of three circles with centers at A, B, and C respectively in order that those whose centers are at B, C shall be tangent to each other externally and shall each be tangent internally to the one whose center is at A?



27. In the triangle ABC, $AB = c$, $BC = a$, $CA = b$. What must be the radii of three circles with centers at A, B, and C respectively in order that those whose centers are at B and C shall be tangent to each other externally and shall each be tangent internally to the one whose center is at A?

CHAPTER VI

FRACTIONAL AND NEGATIVE EXPONENTS. RADICALS

54. The fundamental laws of exponents. — The fundamental laws in the use of positive integral exponents are expressed by the formulæ:

I. $a^m \cdot a^n = a^{m+n}$.

II. $a^m \div a^n = a^{m-n}$. (When n is less than m.)

III. $(a^m)^n = a^{mn}$.

IV. $(ab)^m = a^m b^m$.

V. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

55. Fractional and negative exponents. — Now heretofore we have considered only positive integral exponents, since the purpose of an exponent was to indicate how many times a number was to be taken as a factor. It turns out however to be possible to use fractional and negative exponents in a simple and natural way that is also consistent with these laws. For example, if we agree that $a^{\frac{1}{2}}$ shall be a symbol for a number such that

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a,$$

we shall have a definition that is consistent with the first of these laws. But this is equivalent to saying that $a^{\frac{1}{2}}$ is one of two equal factors of a. We accordingly agree that this symbol shall represent the same number as $\sqrt{a}$.

More generally, if p and q are any two positive integers and we agree that $a^{\frac{p}{q}}$ shall be a symbol for a number such that

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots \text{(to } q \text{ factors)} = a^p,$$

we shall have a definition that is consistent with the first law. But this makes $a^{\frac{p}{q}}$ one of q equal factors of a^p, or a qth root of a^p. We agree then that $a^{\frac{p}{q}}$ shall be a symbol for a number such that

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

It would be in accordance with the fundamental laws of exponents to agree that $9^{\frac{1}{2}} = -3$. But in view of the fact that a positive number has only one positive even root and that any real number has only one real odd root it is agreed to call the positive even root of a positive number and the real odd root of a negative number **principal roots.** Radical signs and fractional exponents are used to indicate these principal roots. Thus, we say that $9^{\frac{1}{2}} = \sqrt{9} = 3$, and $-9^{\frac{1}{2}} = -\sqrt{9} = -3$.

It should be noted that these remarks do not apply to even roots of negative numbers. (See § 133).

56. Suppose now that $x = \sqrt[q]{a}$,

and $y = \sqrt[q]{b}$.

Then $x^q = a$,

and $y^q = b$.

Hence, $x^q y^q = ab$,

and $(xy)^q = ab$. (Law IV.)

Therefore, $xy = \sqrt[q]{ab}$,

or $\sqrt[q]{a} \cdot \sqrt[q]{b} = \sqrt[q]{ab}$,

or $a^{\frac{1}{q}} \cdot b^{\frac{1}{q}} = (ab)^{\frac{1}{q}}$.

A direct application of this relation shows that

$$(a^{\frac{1}{q}})^p = (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p} = a^{\frac{p}{q}}.$$

Hence $a^{\frac{p}{q}}$ may be looked upon as the pth power of $a^{\frac{1}{q}}$ as well as the qth root of a^p.

For example,

$$9^{\frac{1}{2}} = \sqrt{9} = 3,\ 2^{\frac{3}{4}} = \sqrt[4]{2^3} = \sqrt[4]{8},\ (a-b)^{\frac{4}{3}} = \sqrt[3]{(a-b)^4} = (\sqrt[3]{a-b})^4.$$

NOTE. — The symbol $a^{\frac{p}{q}}$ is read "a to the power p divided by q," or "a exponent p divided by q."

If a and b are negative and q is even, care must be taken in forming the product $a^{\frac{1}{q}} \cdot b^{\frac{1}{q}}$. For example, we would naturally say that $(-1)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}} = (-1 \cdot -1)^{\frac{1}{2}} = (1)^{\frac{1}{2}} = 1$. But, as we shall see in § 133, we interpret the symbol $(-1)^{\frac{1}{2}}$ in such a way that $(-1)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}} = -1$.

57. If we agree that a^0 shall be a symbol for a number such that

$$a^0 \cdot a^m = a^{0+m} = a^m;$$

that is, if we agree that this symbol shall obey the first law of exponents, we must say that

$$a^0 = \frac{a^m}{a^m} = 1.$$

It is agreed that this relation shall hold for all values of a except 0. We do not give any meaning to the symbol 0^0.

Also, in accordance with the first law of exponents, we agree that a^{-m}(m a positive number) shall be a symbol for a number such that

$$a^{-m} \cdot a^m = a^0 = 1.$$

This is equivalent to saying that a number with a negative exponent or, as we shall say, raised to a negative power, shall be equal to the reciprocal of this number raised to the corresponding positive power,

$$a^{-m} = \frac{1}{a^m}.$$

With this agreement, the value of a fraction will not be affected if any factor is transferred from the numerator to the denominator, or from the denominator to the numerator, provided the sign of its exponent is changed.

Thus,

$$\frac{x^2}{y^3 z^4} = \frac{x^2 y^{-3}}{z^4} = \frac{1}{x^{-2} y^3 z^4}; \quad \frac{(x+2y)^2(x-3y)^5}{(a-b)^3(a+4b)^2} = \frac{(x+2y)^2(x-3y)^5(a-b)^{-3}}{(a+4b)^2}$$

$$= (x+2y)^2(x-3y)^5(a-b)^{-3}(a+4b)^{-2}.$$

The student is cautioned that this applies only to factors of the numerator and of the denominator, and not to separate terms.

Thus, $\frac{x^3+y^3}{z^2}$ is not equal to $\frac{x^3}{y^{-3}z^2}$ or to $\frac{x^3}{y^{-3}+z^2}$.

58. We have agreed upon a use of fractional, zero, and negative exponents that is consistent with the first law of exponents. It can be shown that this use of these exponents is consistent with all five of the fundamental laws of exponents. It is even possible to use irrational exponents in a way consistent with these laws. Hence we can consider these laws, as given in § 54, as applicable to all real exponents.

This use of fractional, zero, and negative exponents merely gives us new ways for indicating operations with which we were already familiar.

If we were to use these exponents in a way not consistent with the fundamental laws of exponents, the usefulness of algebra as a means for getting information would be seriously impaired.

EXERCISES

Write each of the following expressions without the use of the radical sign:

1. $\sqrt{19}$.
2. $\sqrt[5]{24}$.
3. $\sqrt[3]{3^2}$.
4. $6\sqrt{(x+y)^3}$.
5. $\frac{1}{5}\sqrt[5]{x^3y^5z^{-2}}$.
6. $\sqrt{(x-3y)^3(3x+4y)}$.
7. $\sqrt[3]{a^4+6a^3b-5a^2b^2-2ab^3+7b^4}$.
8. $\frac{1}{3b}\sqrt[3]{\frac{a^4}{b^2}}$.
9. $\sqrt[4]{\frac{(4a+3b)^3(2a-b)^4}{(x+y)^5(3x-2y)^6}}$.
10. $\sqrt{\frac{19}{20}}$.

Write each of the following expressions without the use of fractional or negative exponents:

11. $\dfrac{3}{2^{-\frac{1}{2}}}$.

12. $x^{\frac{2}{3}}y^{-\frac{1}{3}}$.

13. $3^{\frac{1}{2}}x^{\frac{4}{5}}$.

14. $8^{-\frac{1}{3}}$.

15. $(-\frac{4}{5})^{-\frac{4}{5}}$.

16. $5^{\frac{1}{4}}x^{\frac{2}{3}}y^{\frac{7}{8}}z^{-\frac{2}{7}}$.

17. $(\frac{2}{11})^{\frac{1}{3}}$.

18. $(x^3+2x^2y-5xy^2+3y^3)^{\frac{1}{3}}$.

19. $\left(\dfrac{1}{a^2+4ab+8b^2}\right)^{-\frac{4}{5}}$.

20. $(2x-3y)^{\frac{1}{4}}(3x-2y)^{\frac{2}{3}}(2x+3y)^{-\frac{1}{2}}$.

Write each of the following expressions without negative exponents and in as simple a form as possible:

21. $(\frac{36}{49})^{\frac{1}{2}}$.

22. $(.64)^{-\frac{1}{2}}$.

23. $(\frac{2}{5})^{-3}$.

24. 8456^0.

25. $\dfrac{a+a^{-1}}{a-a^{-1}}$.

26. $\dfrac{2-2^{-2}}{2+2^{-2}}$.

27. $\dfrac{1}{x^{-3}+y^{-3}}$.

28. $7x^{-4}$.

29. $(7x)^{-4}$.

30. $(64\,a^{-9}b^6c^{12})^{-\frac{2}{3}}$.

31. $\dfrac{e^x+e^{-x}}{2}$.

32. $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

33. $1+\dfrac{(e^x-e^{-x})^2}{4}$.

Multiply:

34. $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

35. $\sqrt{x^5}+\sqrt{y^5}$ by $x^{\frac{5}{2}}-y^{\frac{5}{2}}$.

36. $m^{\frac{8}{3}}+m^{\frac{4}{3}}n^{\frac{4}{3}}+n^{\frac{8}{3}}$ by $m^{\frac{4}{3}}-n^{\frac{4}{3}}$.

Divide:

37. $a^{\frac{3}{4}} + b^{\frac{3}{2}}$ by $a^{\frac{1}{4}} + b^{\frac{1}{2}}$.

38. $16\,m^2 - 81\,n^2$ by $2\,m^{\frac{1}{2}} - 3\,n^{\frac{1}{2}}$.

39. $15\,a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{2}} - 6\,b$ by $3\sqrt[4]{a} - 2\sqrt{b}$.

Expand:

40. $(a^{\frac{3}{2}} - x^{\frac{1}{2}})^2$.

41. $(a^{\frac{2}{3}} - x^{\frac{2}{3}})^3$.

59. Radicals. — A **radical** is any algebraic expression of the form $\sqrt[n]{a}$, or $b\sqrt[n]{a}$, where n is a positive integer. In such expressions a is called the **radicand**, n the **index**, and b the **coefficient.**

A radical is said to be in its simplest form when:

(a) The radicand is an integer, or a polynomial with integral coefficients if it contains any literal terms.

(b) The radicand contains no factors of the type indicated in (a) raised to powers equal to, or greater than, the index of the radical.

(c) The radicand is not a power whose index has a factor in common with the index of the radical.

Thus the radicals $\sqrt{\frac{2}{3}}$, $\sqrt[3]{(a-a)^4}$, and $\sqrt[6]{a^4}$ are not in their simplest form.

60. Simplification of radicals. — Certain radicals can be simplified by means of one or more of the following reductions. But not every radical can be made to assume the simple form described in the preceding article.

(1) **Reduction of a fractional radicand to the integral form.**

This reduction can always be performed as follows: if the radicand is not a single fraction in its lowest terms, put it into this form. Then, if the radical is of index p, make the denominator of the radicand a perfect pth power by multiplying the numerator and the denominator of the fraction by a properly chosen expression. The original radical is equal to

the pth root of the resulting numerator divided by the pth root of the resulting denominator, which is rational.

Thus, $$\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3},$$

and in general, $$\sqrt[p]{\frac{a}{b}} = \sqrt[p]{\frac{ab^{p-1}}{b^p}} = \frac{\sqrt[p]{ab^{p-1}}}{b}.$$ (Fundamental Law V.)

(2) **The removal of factors from the radicand.**

This reduction can be made only when the radicand contains factors to powers equal to, or greater than, the index of the radical. The following examples illustrate the method of procedure and are based upon Fundamental Law IV.

Examples. $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$

$\sqrt[3]{108} = \sqrt[3]{27 \cdot 4} = \sqrt[3]{27} \cdot \sqrt[3]{4} = 3\sqrt[3]{4}.$

$\sqrt[3]{(a-b)^4} = \sqrt[3]{(a-b)^3} \cdot \sqrt[3]{a-b} = (a-b)\sqrt[3]{(a-b)}.$

(3) **The lowering of the index of the radical.**

When the radicand is a power whose index has a factor in common with the index of the radical, the radical is equal to another radical of lower index.

Example. $\sqrt[6]{a^4} = a^{\frac{4}{6}} = a^{\frac{2}{3}} = \sqrt[3]{a^2}.$

In certain problems it is desirable to introduce factors into the radicand or to increase the index of the radical. These reductions are the inverses of (2) and (3) respectively.

Examples. $4\sqrt[3]{2} = \sqrt[3]{64} \cdot \sqrt[3]{2} = \sqrt[3]{128}.$

$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{125}.$

61. Addition and subtraction of radicals. Definition. — Two or more radicals are said to be *similar* if, when simplified, they have the same index and the same radicand.

For example, $2\sqrt[3]{5}$ and $3\sqrt[3]{5}$ are similar, as are also $\sqrt{a^3b}$ and $3\sqrt{ab}$.

An expression involving two or more radicals can sometimes be simplified by first simplifying each radical and then combining the similar radicals in the way illustrated in the following examples:

Examples. $2\sqrt{98} - 3\sqrt{50} + \sqrt{72} = 14\sqrt{2} - 15\sqrt{2} + 6\sqrt{2} = 5\sqrt{2}.$

$$2\sqrt{98} - 50\sqrt{3} + \sqrt{32} - \sqrt{108} = 14\sqrt{2} - 50\sqrt{3} + 4\sqrt{2} - 6\sqrt{3}$$
$$= 18\sqrt{2} - 56\sqrt{3}.$$

It should be observed that the sum or difference of two dissimilar radicals cannot be expressed as a single radical.

62. Multiplication of radicals. — The product of two radicals with a common index is a radical with the same index whose coefficient and radicand are equal respectively to the products of the coefficients and of the radicands of the factors. (Fundamental Law IV; see § 56.)

Thus, $$a\sqrt[n]{b} \cdot c\sqrt[n]{d} = ac\sqrt[n]{bd}.$$

If the radicals do not have the same index, they should first be reduced to equal radicals with a common index.

Thus, $$\sqrt[3]{5} \cdot \sqrt{6} = \sqrt[6]{25} \cdot \sqrt[6]{216} = \sqrt[6]{5400}.$$

The product of such expressions as $\sqrt{a} + \sqrt{b} - \sqrt{c}$ and $\sqrt{a} - \sqrt{b} + \sqrt{c}$ can be found by applying the usual rules for the multiplication of polynomials in connection with the principles just stated.

Thus,

$$\begin{array}{r} \sqrt{a} + \sqrt{b} - \sqrt{c} \\ \sqrt{a} - \sqrt{b} + \sqrt{c} \\ \hline a + \sqrt{ab} - \sqrt{ac} \\ \quad -\sqrt{ab} \qquad\quad - b + \sqrt{bc} \\ \qquad\quad +\sqrt{ac} \qquad\quad +\sqrt{bc} - c \\ \hline a - b \qquad\qquad\quad + 2\sqrt{bc} - c \end{array}$$

It is, in general, best to simplify all the radicals in a problem before attempting to perform any operations with them.

63. Division of radicals. — The quotient of two radicals with a common index is a radical with this same index whose coefficient and radicand are equal respectively to the quotients of the coefficients and of the radicands of the dividend and the divisor. (Fundamental Law V.)

Thus, $$a\sqrt[n]{b} \div c\sqrt[n]{d} = \frac{a}{c}\sqrt[n]{\frac{b}{d}}.$$

If the radicals do not have the same index, they should first be reduced to equal radicals with a common index.

64. Rationalizing the denominator. — An algebraic expression in the form of the indicated quotient of two polynomials is called a **rational function** of the letters contained in it. If the polynomial in the denominator does not contain any of the letters, the expression is called a **rational integral function** of the letters.

Thus, $\frac{x^3 + 3x^2y + 2y^3}{4x - 5y}$ is a rational function of x and y, while $7x^2 - 6xy + 2y^2$ and $\frac{x^3 + 4y^3}{\sqrt{2}}$ are rational integral functions.

When we say that an expression containing no letters is rational, we mean that it is a rational number; that is, the quotient of two integers.

Certain fractions whose denominators are irrational can be changed into equal fractions with rational denominators.

For example,

$$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{3\sqrt{2}}{2}. \quad \text{(See § 60, 1.)}$$

$$\frac{a}{\sqrt{a} + \sqrt{b}} = \frac{a}{\sqrt{a} + \sqrt{b}} \cdot \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{a(\sqrt{a} - \sqrt{b})}{a - b}.$$

Such a change can always be made when the denominator of the fraction is a single radical or the sum of two terms, one of which is a square root and the other either a square root or

rational. Of course such a change is possible in many other cases also.

If the product of two irrational expressions is rational, either of the expressions is called a **rationalizing factor** of the other.

In computing the numerical value of fractions with irrational denominators it is best first to rationalize the denominators whenever this is possible.

EXERCISES

Simplify the following:

1. $\sqrt{18}$.
2. $\sqrt[3]{\frac{2}{3}}$.
3. $\sqrt{45}$.
4. $\sqrt[4]{\frac{5}{6}}$.
5. $3\sqrt{(x-y)^3}$.
6. $\sqrt[3]{8(a+b)^6}$.
7. $a^{\frac{3}{4}}b^{\frac{5}{4}}$.
8. $\sqrt{a^3+3a^2b+3ab^2+b^3}$.
9. $\sqrt{\frac{a}{b}}$.
10. $\sqrt[3]{\frac{a}{b}}$.
11. $3\sqrt{\frac{1}{2}}-\sqrt{50}+4\sqrt{162}$.
12. $3x^{\frac{5}{3}}-2x^{\frac{2}{3}}$.
13. $3\sqrt{80}-2\sqrt{75}+\sqrt{108}+5\sqrt{20}$.
14. $3\sqrt{a^3}-2\sqrt{ab^2}+4\sqrt{a^5c^2}$.
15. $3\sqrt{1\frac{2}{5}}-5\sqrt{\frac{4}{15}}-2\sqrt{\frac{5}{8}}$.
16. $\sqrt[3]{(a^2-b^2)(a^2+2ab+b^2)}+\sqrt[3]{a-b}$.
17. Which is the greater, $\sqrt[3]{9}$ or $\sqrt{5}$?

$$\sqrt[3]{9}=9^{\frac{2}{6}}=\sqrt[6]{81},$$
$$\sqrt{5}=5^{\frac{3}{6}}=\sqrt[6]{125}.$$

Hence $\sqrt{5}>\sqrt[3]{9}$.

18. Which is the greater, $\sqrt{10}$ or $\sqrt[3]{28}$? $\sqrt{3}$ or $\sqrt[3]{6}$? $\sqrt{19}$ or $\sqrt[3]{65}$?

Perform the following indicated multiplications:

19. $3\sqrt{45}\cdot 4\sqrt{72}$.
20. $6ax^{\frac{2}{3}}\cdot 5ax^{-\frac{1}{2}}$.
21. $\sqrt[5]{\frac{3}{64}}\cdot\sqrt[5]{\frac{2}{7}}$.
22. $\sqrt{(a-b)^3}(\sqrt{a+b})^3$.

23. $(\sqrt{a}-\sqrt{a-b})^2$. 24. $(10-\sqrt{5})(5+\sqrt{10})$.

25. $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}-\sqrt{5})$.

26. $(\sqrt[3]{2}+\sqrt[3]{a})(\sqrt[3]{4}-\sqrt[3]{2a}+\sqrt[3]{a^2})$.

27. Is $1-\sqrt{3}$ a root of the equation $x^2-x-1=0$?

28. Is $\dfrac{-3+\sqrt{5}}{2}$ a root of the equation $x^2+3x+1=0$?

Simplify each of the following expressions:

29. $\sqrt{35}\div\sqrt{\frac{5}{7}}$.

30. $6\div\sqrt{2}$.

31. $5\div\sqrt[3]{4}$.

32. $2\sqrt[n]{a}\div\sqrt[n]{2^n a}$.

33. $\sqrt[3]{a^2bc^2}\div\sqrt[3]{a^2b^2c}$.

34. $(\sqrt{8}+5\sqrt{12}-3\sqrt{14})\div 2$.

35. $(\sqrt{2}+\sqrt{3}-\sqrt{5})\div\sqrt{6}$.

36. $$\frac{\dfrac{\sqrt{1+x^2}-x^2(1+x^2)^{-\frac{1}{2}}}{1+x^2}}{\sqrt{1-\left(\dfrac{x}{\sqrt{1+x^2}}\right)^2}}.$$

37. $$\frac{x(1-x^2)^{-\frac{3}{2}}}{\dfrac{1}{\sqrt{1-x^2}}\sqrt{\left(\dfrac{1}{\sqrt{1-x^2}}\right)^2-1}}.$$

38. $$\frac{x}{\sqrt{1+x^2}-1}\cdot\frac{\sqrt{1+x^2}-1-\dfrac{x^2}{\sqrt{1+x^2}}}{x^2}.$$

39. $$\frac{x}{\sqrt{a-x}-\sqrt{a+x}}.$$

Reduce each of the following fractions to an equal fraction with a rational denominator:

40. $\dfrac{4}{\sqrt{5}-2}$. 41. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$. 42. $\dfrac{1+\sqrt{6}}{1-\sqrt{6}}$.

43. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$. 44. $\dfrac{3}{\sqrt[3]{x}-1}$.

CHAPTER VII

QUADRATICS

65. Quadratic equations. — Sometimes our information concerning a number comes to us in the form of an equation of the second degree in the unknown. Such an equation is known as a **quadratic** equation. In this chapter we shall consider how to identify a number described by such an equation.

Consider the equation $6x^2 - x - 2 = 0.$ (1)

We observe that the left member can be factored and that the equation can be written in the form

$$(3x - 2)(2x + 1) = 0. \tag{2}$$

Now we know from Theorem 10, Chapter I, that if x is any number that satisfies this equation, it must be such that either

$$3x - 2 = 0, \tag{3}$$

or

$$2x + 1 = 0; \tag{4}$$

and conversely, any number of this kind must satisfy the given equation. That is, the numbers described by (1) are the same as the numbers described by (3) and (4).

In general, if the right member of an equation in one unknown is zero and the left member is factored, any number which when put in place of the unknown makes one of these factors zero must satisfy the equation, and, conversely, any root of the equation when put in place of the unknown must make one of these factors zero. Hence,

66. *In order to solve an equation in one unknown, transpose all the terms to the left member of the equation, making the right member zero; factor the resulting left member; put equal to zero*

each of these factors that contains the unknown and solve the resulting equations, which will be of lower degree than the original equation.

This method of solving an equation is an important one and is applicable to equations of any degree. In practice, however, it is subject to limitations because there are so many polynomials in one variable whose factors we cannot find.

67. Equations with given roots. — With Theorem 10 of Chapter I in mind it is easy to form an equation that shall have given numbers for roots.

If, for example, the given numbers are $r_1, r_2, \cdots r_n$, it is obvious that

$$(x - r_1)(x - r_2) \cdots (x - r_n) = 0$$

is an equation whose roots are these numbers.

EXERCISES

Solve the following equations by factoring, and check your results:

1. $x^2 - 1 = 0$.
2. $x^2 - 2x + 1 = 0$.
3. $x^3 - 2x^2 + x = 0$.
4. $x^2 + 9x + 8 = 0$.
5. $x^4 - 13x^2 + 36 = 0$.
6. $x^3 + 6x^2 + 11x + 6 = 0$.
7. $4x(x - 5)(x + 7) = 0$.
8. $y^2 = 25$.
9. $y^2 - 20 = y$.
10. $2x^2 - 11x + 12 = 0$.
11. $5z^2 + 29z - 6 = 0$.
12. $0 = x + 4x^2$.
13. $0 = m^2 + 5m + 6$.
14. $x^3 + 3x^2 - 33x - 35 = 0$.
15. $x^3 + 6x^2 + 2x = 30 + 3x$.
16. $x^2 + 25x + 144 = 0$.

Form an equation whose roots are:

17. 2, 3.
18. -1, 5.
19. $\sqrt{2}$, $-\sqrt{2}$.
20. -1, 2, 4.
21. $\frac{2}{3}$, 1, $\frac{3}{5}$.
22. $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$.
23. $\frac{-3+\sqrt{41}}{4}$, $\frac{-3-\sqrt{41}}{4}$.

68. Solution by completing the square. — When we have difficulty in applying this method to the solution of a quadratic equation we can proceed as illustrated in the following examples:

(a) Solve the equation

$$3x^2 + 2x = 4. \tag{1}$$

Divide each member by 3,

$$x^2 + \tfrac{2}{3}x = \tfrac{4}{3}. \tag{2}$$

Add $(\frac{1}{3})^2$ to each member,

$$x^2 + \tfrac{2}{3}x + \tfrac{1}{9} = \tfrac{13}{9}. \tag{3}$$

Extract the square root of each member,

$$x + \tfrac{1}{3} = \pm\frac{\sqrt{13}}{3},$$

$$x = \frac{-1 \pm \sqrt{13}}{3}.$$

Check.

$$3\left(\frac{-1 \pm \sqrt{13}}{3}\right)^2 + 2 \cdot \frac{-1 \pm \sqrt{13}}{3}$$

$$= \frac{1 \mp 2\sqrt{13} + 13}{3} + \frac{-2 \pm 2\sqrt{13}}{3}$$

$$= \tfrac{12}{3} = 4.$$

(b) Solve the equation

$$3x^2 + 2x + 4 = 0.$$

Transpose 4 to the left member,

$$3x^2 + 2x = -4.$$

Divide each member by 3,

$$x^2 + \tfrac{2}{3}x = -\tfrac{4}{3}.$$

Add $(\frac{1}{3})^2$ to each member,

$$x^2 + \tfrac{2}{3}x + \tfrac{1}{9} = -\tfrac{11}{9}.$$

Extract the square root of each member,

$$x + \frac{1}{3} = \pm\frac{\sqrt{-11}}{3},$$

$$x = \frac{-1 \pm \sqrt{-11}}{3}.$$

Check. $$3\left(\frac{-1\pm\sqrt{-11}}{3}\right)^2+2\cdot\frac{-1\pm\sqrt{-11}}{3}+4$$
$$=\frac{1\mp2\sqrt{-11}-11}{3}+\frac{-2\pm2\sqrt{-11}}{3}+\frac{12}{3}$$
$$=0.$$

In simplifying $\left(\frac{-1\pm\sqrt{-11}}{3}\right)^2$ the student should bear in mind that $\sqrt{-11}\cdot\sqrt{-11}=-11$. (See § 133.)

The terms involving the unknown x should be brought to the left side and the constant terms to the right side of the equation. If the coefficient of x^2 is not a perfect square, it should be made such by multiplying or dividing each member of the equation by a properly chosen constant. The term in x should then be divided by twice the square root of the term in x^2, and the square of the quotient added to each member of the equation. The resulting left member will be a trinomial square. Why?

The roots of the equation can now be found by putting one of the square roots of the left member equal in turn to each of the square roots of the right member and solving each of the resulting linear equations.

This method of solving a quadratic equation is known as the *method of solution by completing the square.*

69. Equation (3) of the first illustrative example can be written thus
$$x^2+\tfrac{2}{3}x+\tfrac{1}{9}-\tfrac{13}{9}=0.$$

The left member of this equation can be looked upon as the difference of two squares and factored accordingly.

Thus, $$\left(x+\frac{1}{3}+\frac{\sqrt{13}}{3}\right)\left(x+\frac{1}{3}-\frac{\sqrt{13}}{3}\right)=0.$$

Equating each of these factors to zero, and solving the resulting equations, we get
$$x=\frac{-1-\sqrt{13}}{3}\text{ and }x=\frac{-1+\sqrt{13}}{3},$$
as before.

Thus the operation of completing the square in a quadratic equation may be considered merely as a device for enabling us to see the factors of the left member of the equation when the right member is zero. And therefore the method of solution by completing the square is essentially a part of the method of solution by factoring. It should be observed, however, that factors obtained in this way are not always of the kind considered in Chapter II. (See § 18.)

EXERCISES

Solve each of the following equations by completing the square, and check your results:

1. $3x^2 = 2x + 5$.
2. $2x^2 + 5x - 3 = 0$.
3. $5x^2 + 3x = 3$.
4. $7y + 2 = 5y^2$.
5. $4t^2 = t + 3$.
6. $\frac{x^2}{4} + \frac{(3x+b)^2}{9} = 1$.
7. $4r^2 + 6r + 1 = 0$.
8. $2y^2 + y - 3 = 0$.
9. $x^2 = 3 - x$.
10. $(mx+3)^2 = 4x$.
11. $x^2 + x + 1 = 0$.
12. $x^2 - x + 1 = 0$.
13. $(2x-3)^2 + (x-1)^2 = 3x + 4$.
14. $lx^2 + mx + n = 0$.
15. $6z^2 - 13z + 6 = 0$.
16. $x^2 = 4$.
17. $5x^2 + 13 = 0$.
18. $x^2 = 7x$.
19. $3y^2 - 2 = 0$.
20. $3r^2 - r = 0$.
21. $x^2 + x - 1 = 0$.
22. $x^2 - x - 1 = 0$.
23. $\frac{x^2}{9} + \frac{(x-1)^2}{4} = 1$.
24. $x^2 + 1 = 0$.
25. $(3x+1)^2 = 8x$.
26. $5y^2 + 5y + 1 = 0$.
27. $(x+1)^3 - x^3 = 1$.
28. $x^2 + ax = 2$.
29. $(k+1)y^2 + ky + 3 = 0$.
30. $\frac{3x^2}{2} - \frac{2(x-2)^2}{3} = 1$.

70. Solution by formula. — The equation

$$ax^2 + bx + c = 0$$

is called the *general quadratic equation*, since it reduces to any given quadratic when proper values are given to a, b, and c. The solution of this general quadratic will therefore give us a formula from which we can write down the roots of any quadratic.

Now the student can verify by using the method of completing the square that

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the roots of this equation.

Before this formula is applied to the solution of a given equation, the equation must be simplified and all its terms brought into the left member.

Suppose for example that we want to solve the equation

$$\frac{2}{x} + \frac{1}{x+2} = \frac{2}{5}.$$

Clear of fractions, $10x + 20 + 5x = 2x^2 + 4x.$

Transpose all the terms to the left member,

$$-2x^2 + 11x + 20 = 0.$$

Then by the formula ($a = -2$, $b = 11$, $c = 20$)

$$x = \frac{-11 \pm \sqrt{121 + 160}}{-4} = \frac{-11 \pm \sqrt{281}}{-4}.$$

Check.

$$\frac{2}{\frac{-11 \pm \sqrt{281}}{-4}} + \frac{1}{\frac{-11 \pm \sqrt{281}}{-4} + 2}$$

$$= \frac{-8}{-11 \pm \sqrt{281}} + \frac{-4}{-19 \pm \sqrt{281}}$$

$$= \frac{152 \mp 8\sqrt{281} + 44 \mp 4\sqrt{281}}{(-11 \pm \sqrt{281})(-19 \pm \sqrt{281})}$$

$$= \frac{196 \mp 12\sqrt{281}}{490 \mp 30\sqrt{281}}$$

$$= \frac{2(98 \mp 6\sqrt{281})}{5(98 \mp 6\sqrt{281})} = \frac{2}{5}.$$

EXERCISES

Solve the following quadratics by means of the formula:

1. $x^2+3x+1=0.$
2. $2y^2-5y-3=0.$
3. $x^2+7x=5-4x^2.$
4. $\frac{x+1}{3}-\frac{x^2-5}{4}=10.$
5. $lx^2+mx+n=0.$
6. $mx^2-3x+c=0.$
7. $x^2+ax+b=0.$
8. $3x+6x\frac{7+x}{2}-8\left(\frac{7+x}{2}\right)^2+3x^2=0.$
9. $4y^2-5y=0.$
10. $m^2-6m+5=0.$
11. $(8m-6)^2=256m^2.$
12. $kx^2-3(k+1)x+5=0.$
13. $9(k+1)^2=20k.$
14. $(y-1)^2=3(2y-3).$
15. $64m^2-256(1+m^2)=0.$
16. $(mx+k)^2=6x.$
17. $x^2+\left(\frac{7-x}{3}\right)^2=4.$
18. $\frac{(x+5)^2}{9}+\frac{(3x-4)^2}{16}=1.$
19. $17y+4=8y^2.$
20. $x^2-16=0.$
21. $(5y+2)^2+y^2=\frac{5}{3}.$
22. $bx^2+bx=1.$
23. $(x-2)(7x-1)=5.$
24. $(x-2)(mx-1)=5.$
25. $x^2+m^2x^2+8mx+64=9.$
26. $m^2x^2+8mx+64=6x.$
27. $\frac{x^2}{16}+\frac{(3x+b)^2}{9}=1.$
28. $\frac{x^2}{16}-\frac{(3x+5)^2}{9}=1.$
29. $6x^2+\frac{(b-2x)^2}{9}=12.$
30. $x^2-3x-m(x+2x^2+4)=5x^2+3.$
31. Show that

$$ax^2+bx+c=a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right).$$

71. Nature of the roots of a quadratic. — Certain things about the roots of a quadratic equation can be determined without solving the equation. If the equation is in the form

$$ax^2+bx+c=0,$$

the roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hence, when a, b, and c are rational numbers,

I. *If $b^2 - 4ac = 0$, there is only one root; namely, $-\dfrac{b}{2a}$.*

In this case it is customary to say that there are *two equal* roots.

II. *If $b^2 - 4ac$ is the square of a rational number different from zero, the roots are real, unequal, and rational.*

III. *If $b^2 - 4ac$ is positive and not the square of a rational number, the roots are real, unequal, and irrational.*

IV. *If $b^2 - 4ac$ is negative, the roots are imaginary and unequal.*

The expression $b^2 - 4ac$ is called the **discriminant** of the equation.

Thus the roots of the equation

$$3x^2 + 5x + 1 = 0$$

are real, distinct, and irrational, since its discriminant equals 13.

If we call the two roots of the equation

$$ax^2 + bx + c = 0$$

x_1 and x_2 respectively, so that

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

then $x_1 + x_2 = -\dfrac{b}{a}$, and $x_1 x_2 = \dfrac{b^2 - (b^2 - 4ac)}{4a^2} = \dfrac{c}{a}$.

Hence, *the sum of the roots of the equation*

$$ax^2 + bx + c = 0$$

is equal to minus the coefficient of x divided by the coefficient of x^2, and the product of the roots is equal to the constant term divided by the coefficient of x^2. (See § 150.)

For example, the sum of the roots of the equation

$$3x^2 + 5x + 1 = 0$$

is $-\frac{5}{3}$ and the product is $\frac{1}{3}$.

The student should verify this statement by solving the equation and forming the sum and the product of the roots.

If the equation is not in the standard form, it must be reduced to this form before this rule is applied.

Consider, for example, the equation

$$3x^2 + 1 = 5x - 4x^2.$$

Transposing, $$7x^2 - 5x + 1 = 0.$$

Hence, the sum of the roots is $\frac{5}{7}$ and the product is $\frac{1}{7}$.

EXERCISES

Determine without solving the equations the nature of the roots of each of the following equations. Also find the sum and the product of the roots of each equation:

1. $2x^2 + 7x + 1 = 0.$
2. $x^2 + \frac{(x-4)^2}{9} = 16.$
3. $x^2 - x + 2 = 0.$
4. $x^2 - x - 2 = 0.$
5. $(3x + 5)^2 = 6x.$
6. $\frac{x^2}{4} + \frac{(x+3)^2}{9} = 1.$
7. $5x^2 + 5x = 1.$
8. $x^2 = 10x - 25.$
9. $5x^2 + 5x = 0.$
10. $x^2 + 6x + 9 = 0.$
11. $(\frac{3}{2}x + 1)^2 = 5(x + 2).$
12. $(2 + x)^2 = (1 + 2x)^2.$
13. $4m^2 - 6m + 5 = 0.$
14. $r^2 + 10r = 0.$
15. $m^2 + 7 = 0.$
16. $x(-2x + 3) = 5.$
17. $x^2 + 2x(2x - 1) + (2x - 1)^2 + x + (2x - 1) + 1 = 0.$
18. $(-x + 2)^2 + 6(-x + 2) + 9 = 7(x - 3).$
19. $x^2 + 3x + 9 = 0.$
20. $x^2 - 3x + 9 = 0.$

For what values of k will the roots of each of the following equations be equal?

21. $x^2 - 4kx + 4 = 0.$

Here $a = 1$, $b = -4k$, and $c = 4$. Hence, $b^2 - 4ac = 16k^2 - 16$. In order that the roots be equal, we must have

$$16k^2 - 16 = 0.$$

Solving this equation for k we get $k = \pm 1$. That is, when k in this equation has the value 1, or -1, the equation has equal roots. We can readily verify this result. When $k = 1$, the equation is

$$x - 4x + 4 = 0$$

and 2 is the only root; when $k = -1$, the equation is

$$x + 4x + 4 = 0$$

and -2 is the only root.

22. $x^2 - 6kx + 12 = 0$.
23. $x^2 - kx + 1 = 0$.
24. $10x^2 + 6kx + k^2 - 4 = 0$.
25. $x^2 + 2kx + k^2 = 4x$.
26. $9x^2 + 6kx + k^2 = 10x$.
27. $4x^2 + 4kx + 4k^2 = 3(x - 1)$.
28. $k^2x^2 + 2kx + 1 = 2x$.
29. $(kx + 3)^2 = 4(x + 2)$.
30. $(2x + k)^2 = 3(x - 5)$.
31. $\dfrac{k^2(y + \frac{1}{2})^2}{9} + \dfrac{y^2}{16} = 1$.
32. $3x^2 + 5kx(x - 5) - 2k^2(x - 5)^2 + 5x + 3k(x - 5) + 4 = 0$.

72. Graphical solution of the quadratic equation. — Suppose we draw the locus or graph of the equation

$$y = x^2.$$

The first thing to do is to find pairs of values of x and y that satisfy this equation. This can be done by taking values of x at will and computing the corresponding values of y. Some of the results are given in the following table:

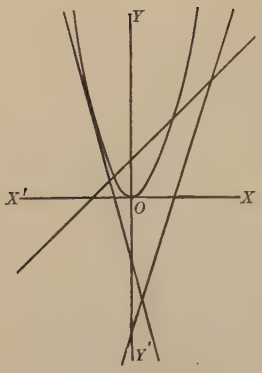
x	-5	-4	-3	-2	-1	$-\frac{1}{2}$	0	1	2	$\frac{9}{4}$	3	4	5
y	25	16	9	4	1	$\frac{1}{4}$	0	1	4	$\frac{81}{16}$	9	16	25

If we plot the points whose coördinates are these corresponding pairs of values of x and y and connect these points by a smooth curve, we shall have an approximate representation of the locus.

If now we draw the locus of the equation

$$y = x + 2$$

with reference to the same set of axes, the two loci will intersect in two points. The coördinates of these two points must satisfy both equations,



and hence the abscissæ are values of x that make the right members of these equations equal; that is, for values of x equal to these abscissæ

$$x^2 = x + 2,$$

or $$x^2 - x - 2 = 0.$$

These abscissæ are therefore the roots of this last equation.

These considerations suggest a general method for solving a quadratic equation graphically. If the equation is

$$ax^2 + bx + c = 0,$$

or

$$x^2 = -\frac{b}{a}x - \frac{c}{a},$$

we draw the loci of the two equations

$$y = x^2,$$

and

$$y = -\frac{b}{a}x - \frac{c}{a},$$

and measure the abscissæ of their points of intersection. These abscissæ are the roots of the given equation.

The locus of the first equation is called a **parabola**. The locus of the second equation is evidently a straight line.

If the line and the parabola have two points in common, the quadratic has two real roots.

If the line and the parabola have only one point in common, the quadratic has only one root.

If the line and the parabola have no points in common, the roots of the quadratic are imaginary. In this case this method gives us no information concerning the roots beyond the fact that they are imaginary.

73. Find the roots of the equation

$$x^2 - 3x + 7 = 0.$$

We have the locus of the equation

$$y = x^2$$

already drawn. If we draw the locus of

$$y = 3x - 7,$$

we find that the two loci have no points in common and we conclude that the roots of the quadratic are imaginary. That this conclusion is correct can be verified directly by consideration of the value of the discriminant of the given equation.

Find the roots of the equation

$$x^2 + 4x + 4 = 0.$$

Using the same figure for the locus of the equation

$$y = x^2,$$

and drawing the locus of the equation

$$y = -4x - 4,$$

we find that the two loci have only one point in common. The abscissa of this point is -2. Hence -2 is the only root of the given equation.

The student must not expect to get exact solutions by graphical methods.

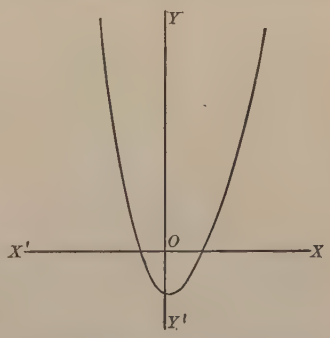
74. Second graphical method. — There is another graphical method for the solution of a quadratic that is in common use. It is sufficiently explained by the following example.

Suppose that we wish to find the roots of the equation

$$x^2 - x - 2 = 0.$$

Draw the locus of the equation

$$y = x^2 - x - 2.$$



In order to do this we take values of x at will and compute from the equation the corresponding values of y. Then the smooth curve drawn through the points whose coördinates are these corresponding pairs of values of x and y is the approximate locus of the equation.

If this locus has a point in common with the x-axis, the abscissa of such a point is a value of x that makes

$$x^2 - x - 2 = 0,$$

that is, is a root of this equation, since the ordinate of every point on the x-axis is 0.

In general, in order to solve the equation

$$ax^2 + bx + c = 0$$

by this method, plot the locus of the equation

$$y = ax^2 + bx + c,$$

and measure the abscissæ of the points this locus has in common with the x-axis. These abscissæ are the roots of the given equation.

EXERCISES

Solve each of the following equations graphically.

Do not use the same method for all of the equations.

1. $x^2 + x - 1 = 0.$
2. $x^2 - x - 1 = 0.$
3. $2x^2 + x - 2 = 0.$
4. $x^2 + 3x + 1 = 0.$
5. $5x^2 + 7x - 2 = 0.$
6. $5x^2 + 7x + 2 = 0.$
7. $2x^2 + 8x + 8 = 0.$
8. $x^2 + 6x + 10 = 0.$
9. $3x^2 + 8x + 5 = 0.$
10. $8x^2 + 3x - 5 = 0.$
11. $x^2 - 8x + 16 = 0.$
12. $x^2 - 8x + 20 = 0.$
13. $x^2 - 8x + 6 = 0.$
14. $3x^2 = 0.$
15. $x^2 = 1.$
16. $x^2 + 5x + 1 = 0.$
17. $x^2 + 3x + 2 = 0.$
18. $(x + 4)^2 = 9.$
19. $(3x - 8)^2 = 0.$
20. $3x^2 + 2x + 1 = 0.$

75. Equations involving fractions. — When we clear an equation of fractions by multiplying each member by the lowest common denominator of the fractions, the resulting

equation may have roots that are not roots of the original equation. In other words, the new equation may not describe the same numbers as are described by the original equation.

Consider, for example, the equation

$$\frac{2x}{x^2-9} = \frac{1}{x-3} + 4.$$

Clearing of fractions, $2x = x + 3 + 4(x^2 - 9),$

$$(x-3) - 4(x^2-9) = 0,$$

$$(x-3)(1-4x-12) = 0.$$

One of the roots of this equation is 3. But this evidently cannot be a root of the original equation, since for $x = 3$ the left member becomes $\frac{6}{0}$, and this has no numerical value.

76. In many cases the roots of the new equation — that is, the equation resulting from clearing the original equation of fractions — are the same as the roots of the original equation. Some of the circumstances under which this can occur may be inferred from the following theorem:

Theorem. — *Any root of the new equation that is not a root of the original equation must when substituted for x make some denominator of the original equation zero.*

Suppose that the equation has been brought to the form

$$P = 0$$

by transposing all the terms of the right member to the left member. Here P represents some fractional expression.

If D is the lowest common denominator of the fractions occurring in P, the new equation is

$$PD = 0,$$

and PD is a polynomial in x.

If now any value of x makes PD equal to zero and D different from zero, it must make P equal to zero. That is, it must be a root of the original equation. Hence any root of the new equation that is not a root of the original equation must be a root of the equation

$$D = 0.$$

But any root of this last equation must, when substituted for x, make some denominator of the original equation zero. (See 10, Chap. I.)

It follows from this that when no denominator of the original equation contains the unknown, the roots of the new equation must be the same as those of the original equation; that is, the two equations must be equivalent.

77. Hence, after having solved the new equation, the student should determine by trial whether any of its roots, when substituted for x, makes any denominator of the original equation equal to zero. No root of this kind is a root of the original equation. But all the other roots of the new equation are roots of the latter, and every root of the original equation is a root of the new equation.

If we clear the equation of fractions by multiplying each member by a polynomial that is a common denominator not of the lowest degree, the two equations will certainly not be equivalent.

EXERCISES

Solve each of the following equations:

1. $\dfrac{x^2-3x+5}{x^2+3x-5}=4.$

2. $\dfrac{1}{3x^2-5x-2}+\dfrac{2}{x^2-4}=\dfrac{1}{x+2}.$

3. $\dfrac{x^2+5x+6}{x^2-9}=x+4.$

4. $\dfrac{4y}{y^2-4}+\dfrac{2}{y-2}=5.$

5. $\dfrac{3r}{r^3-9}=\dfrac{5}{r+3}+\dfrac{6}{r-3}.$

6. $\dfrac{5}{2(x-3)}=\dfrac{6}{x-2}.$

7. $\dfrac{c-a}{x-a}=3\dfrac{c-b}{x-b}.$

8. $\dfrac{x+5}{x-5}=\dfrac{3+x}{3-x}.$

9. $\dfrac{6}{5-3x}+3x-4=x+\dfrac{4}{3x-5}.$

10. $\dfrac{4x}{x+4}-\dfrac{3x}{x-5}=\dfrac{x-14}{x^2-x-20}.$

11. $.02x^2-3.64x-2.5=0.$

12. $.01x(.3x+4.5)=.34x^2-5.$

13. $\dfrac{1}{x-k}+\dfrac{1}{x-l}+\dfrac{1}{x-m}=0.$

14. $\dfrac{12x}{x^2-9}-\dfrac{x+3}{x-3}=1.$

15. $\dfrac{x-a}{x+a}+\dfrac{x+a}{x-a}=\dfrac{a}{b}.$

16. $\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x-2}=\dfrac{a}{x}.$

17. $\dfrac{3}{x+1}-\dfrac{2}{x-1}=4.$

18. $\dfrac{2x}{x^2-3x+2}+\dfrac{5}{x-2}=1.$

78. Equations involving radicals. — Certain equations involving radicals can be solved by the methods described in the following illustrative examples:

It is understood that in these problems we are to attach to every radical its principal value.

Example 1. Solve the equation

$$\sqrt{x^2-9}-4=0.$$

Transposing -4 to the right member,

$$\sqrt{x^2-9}=4.$$

Squaring both members,

$$x^2-9=16,$$

$$x^2=25,$$

$$x=\pm 5.$$

Check. $\sqrt{25-9}-4=4-4=0.$

In dealing with rational integral equations we checked our results merely for the sake of guarding against error. But in solving an equation with radicals we sometimes get an apparent root that is in reality not a root even when we have proceeded in the regular way. This also happened in the solution of fractional equations. Hence the operation of checking is a necessary part of the solution of such equations. This is illustrated in the next example.

Example 2. Solve the equation

$$\sqrt{x-6}+4=0.$$

Transposing,

$$\sqrt{x-6}=-4.$$

Squaring both members,

$$x-6=16,$$

$$x=22.$$

Check. $\sqrt{22-6}+4=\sqrt{16}+4=4+4\neq 0.$

Hence 22 is not a root of the equation. As a matter of fact, the equation has no root.

The explanation of the appearance of these false roots is as follows:

If the original equation is $L=R$,

the resulting equation is $L^2=R^2$,

or $L^2-R^2=0.$

The roots of this last equation are the roots of the equations

$$L-R=0$$

and $$L+R=0.$$

And if the equation $L+R=0$

has a root, this is also a root of the equation

$$L^2=R^2$$

that is not a root of the equation

$$L=R,$$

provided that it does not make $L=0$ and $R=0$.

In example (1) $L=\sqrt{x^2-9}$ and $R=4$. Here the equation $L+R=0$ has no root, and therefore the equation $L^2=R^2$ is equivalent to the equation $L=R$.

But in example (2) $L=\sqrt{x-6}$ and $R=-4$, and therefore the equation $L+R=0$ has a root; namely, 22. This is not a root of the original equation, since in this case $R\neq 0$ for any value of x.

Example 3. Solve the equation

$$\sqrt{x+1}+\sqrt{x-3}=5.$$

Transposing $\sqrt{x-3}$ to the right member,

$$\sqrt{x+1}=5-\sqrt{x-3}.$$

Squaring both members, $x+1=25-10\sqrt{x-3}+x-3.$

Transposing, $10\sqrt{x-3} = 21.$

Squaring both members,

$$100(x-3) = 441,$$
$$x = 7.41.$$

Check. $\sqrt{7.41+1} + \sqrt{7.41-3} = \sqrt{8.41} + \sqrt{4.41} = 2.9 + 2.1 = 5.$

Example 4. Solve the equation

$$\sqrt{x^2-10x+41} - \sqrt{x^2+10x+41} = 8.$$

Transposing $-\sqrt{x^2+10x+41}$ to the right member,

$$\sqrt{x^2-10x+41} = 8 + \sqrt{x^2+10x+41}.$$

Squaring both members,

$$x^2 - 10x + 41 = 64 + 16\sqrt{x^2+10x+41} + x^2 + 10x + 41,$$
$$-20x - 64 = 16\sqrt{x^2+10x+41},$$
$$5x + 16 = -4\sqrt{x^2+10x+41}.$$

Squaring both members,

$$25x^2 + 160x + 256 = 16x^2 + 160x + 656,$$
$$9x^2 = 400,$$
$$x = \pm \tfrac{20}{3}.$$

Check for $x = \frac{20}{3}$.

$$\sqrt{\tfrac{400}{9} - \tfrac{200}{3} + 41} - \sqrt{\tfrac{400}{9} + \tfrac{200}{3} + 41} = \sqrt{\tfrac{169}{9}} - \sqrt{\tfrac{1369}{9}} = \tfrac{13}{3} - \tfrac{37}{3} = -8.$$

Hence, $\frac{20}{3}$ is not a root of the equation.

Check for $x = -\frac{20}{3}$.

$$\sqrt{\tfrac{400}{9} + \tfrac{200}{3} + 41} - \sqrt{\tfrac{400}{9} - \tfrac{200}{3} + 41} = \tfrac{37}{3} - \tfrac{13}{3} = 8.$$

Hence, $-\frac{20}{3}$ is a root of the equation.

79. The preceding equations contain no radicals except square roots. The most common irrational equations are of this kind and the method of solution illustrated in these examples is usually applicable in such cases. It may be formulated as follows:

If necessary, so transpose the terms that one radical stands alone in one member of the equation.

Then form a new equation by equating the squares of the members of the last equation.

If the resulting equation contains radicals, repeat this process until an equation is obtained that is free from radicals.

Solve this last equation and test each of its roots by substituting it in the original equation.

In simplifying the radicals in order to determine whether a number satisfies the equation, the student must not resort to squaring both members of the equation, since the question he is trying to settle is whether the equation formed in this way is, or is not, equivalent to the original equation.

EXERCISES

Solve the following equations and check your results:

1. $\sqrt{x+2}-x=0.$
2. $\sqrt{2x+5}=\sqrt{x-10}.$
3. $\sqrt{4x+13}+x=2.$
4. $\dfrac{\sqrt{x^2+9}}{3}+\dfrac{x-5}{4}=\dfrac{17}{12}.$
5. $\sqrt{x^2+3x+2}=x+1.$
6. $\dfrac{\sqrt{x^2+16}}{4}+\dfrac{x-9}{5}=2.$
7. $\sqrt{x^2+10x+41}-\sqrt{x^2-10x+41}=8.$
8. $\sqrt{x^2+5x+22}-\sqrt{x^2-5x+22}=2.$
9. $x+\sqrt{x(6+x)}=\frac{9}{2}.$
10. $\sqrt{(x+2)^2+3}+\sqrt{(x-1)^2+3}=5.$
11. $\sqrt{8x+1}=\sqrt{x^2+6x-2}.$
12. $\sqrt{(x-3)^2+7}+\sqrt{(x+7)^2+7}=8.$
13. $\sqrt{5x-6}-\sqrt{x-2}=2.$
14. $\sqrt{3x+10}=\sqrt{x-1}+\sqrt{2x-1}.$

15. $1125 = 1093\sqrt{1 + .003665\,t}$.

16. $v = 1093\sqrt{1 + .003665\,t}$.

Solve for t. This formula gives the velocity of sound in air in feet per second at a temperature of $t°$ Centigrade. The formula in Ex. 15 is a particular case of this.

17. $\sqrt{5 + (6 - x)^2} = 5$.

18. $\sqrt{9 + (4 + x)^2} = \sqrt{(3 + x)^2 + 16}$.

19. $x - 1 - \sqrt{1 + x} = 0$.

20. $2x + 3 = 4 - \sqrt{x}$.

21. $\sqrt{(x-3)^2 + 7} - \sqrt{(x+7)^2 + 7} = 8$.

22. $\sqrt{y^2 - y + 1} + \sqrt{y^2 - 2y + 1} = 1$.

80. Equations in the form of quadratics. — We can frequently apply the methods for solving quadratics to the solution of equations that involve the unknown only in a certain expression and in the square of this expression.

Example 1. Solve the equation

$$3x^4 - 14x^2 + 8 = 0.$$

This can be written $3(x^2)^2 - 14x^2 + 8 = 0$.

Hence,
$$x^2 = \frac{14 \pm \sqrt{196 - 96}}{6} = 4, \text{ or } \frac{2}{3},$$
$$x = \pm 2, \text{ or } \pm\sqrt{\tfrac{2}{3}}.$$

The student should check these results.

Example 2. Solve the equation

$$3x - 5 - 4\sqrt{3x - 5} + 3 = 0.$$

This can be written
$$(\sqrt{3x - 5})^2 - 4\sqrt{3x - 5} + 3 = 0.$$

Hence,
$$\sqrt{3x - 5} = 3, \text{ or } 1,$$
$$3x - 5 = 9, \text{ or } 1,$$
$$x = \tfrac{14}{3}, \text{ or } 2.$$

The student should check these results.

Example 3. Solve the equation

$$x^2 - 3x + 5 - 3\sqrt{x^2 - 3x + 8} = -5.$$

This can be changed into a quadratic in $\sqrt{x^2 - 3x + 8}$ by adding 3 to each member. Thus,

$$x^2 - 3x + 8 - 3\sqrt{x^2 - 3x + 8} = -2.$$

Hence,

$$\sqrt{x^2 - 3x + 8} = 2, \text{ or } 1,$$

$$x^2 - 3x + 8 = 4, \text{ or } 1.$$

$$x = \frac{3 \pm \sqrt{-7}}{2}, \text{ or } \frac{3 \pm \sqrt{-19}}{2}.$$

The student should check these results.

EXERCISES

Solve the following equations:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^6 - 9x^3 + 8 = 0.$
3. $(x^2 + 4x - 1)^2 + x^2 + 4x = 3.$
4. $-6\sqrt{x^2 + 1} + x^2 + 10 = 0.$
5. $\left(x + \frac{3}{x}\right)^2 + 2\left(x + \frac{3}{x}\right) = 15.$
6. $2x^4 - 9x^2 - 5 = 0.$
7. $(x^2 - 1)^2 + 3(x^2 - 1) + 2 = 0.$
8. $x^2 - 4x + 5 - 7\sqrt{x^2 - 4x + 5} + 10 = 0.$
9. $x^6 + 26x^3 - 27 = 0.$
10. $x^4 - 2x^2 + 1 = 0.$
11. $y^4 - 13y^2 + 36 = 0.$
12. $7\sqrt{x^2 + 3x} + 2x^2 + 6x - 4 = 0.$
13. $x^2 + 8x + 1 + 3\sqrt{x^2 + 8x + 2} = 9.$
14. $\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) + 6 = 0.$
15. $(z^2 + 3z)^2 - 2(z^2 + 3z) - 8 = 0.$
16. $2z^4 - 11z^2 + 12 = 0.$
17. $x^6 + 7x^3 - 8 = 0.$
18. $(y^2 - 5)^2 = 16.$
19. $-6\sqrt{x + 1} + x + 10 = 0.$
20. $\left(\frac{1}{x^2} - 1\right)^2 - 3\left(\frac{1}{x^2} - 1\right) + 2 = 0.$

81. The conditions of many problems require that the unknown number be real. If such a problem leads to a quadratic equation and if one of the known numbers is represented by a letter, we can frequently find limitations on the possible values of this known number. The method of procedure is illustrated in the following example.

Find two positive numbers such that their product is 36 and their sum s.

Let $x =$ one of the numbers.

Then $s - x =$ the other one.

Hence $x(s - x) = 36,$

or $x^2 - sx + 36 = 0.$

In order for the roots of this equation to be real we must have $s^2 - 144 \geqq 0$. Hence the least possible value for s is 12.

What are the required numbers for this value of s?

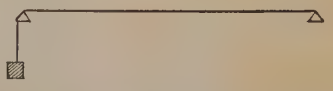
PROBLEMS

In solving a quadratic it is generally best to use the formula, unless the method of factoring can be applied readily.

Formula from physics. — The number n of complete vibrations per second made by a stretched wire is given by the formula

$$n = \frac{1}{2l}\sqrt{\frac{980\,M}{m}},$$

where l is the length in centimeters between the bridges, M the weight in grams of the stretching weight, and m the weight in grams of the wire per centimeter of length.



1. What must be the stretching weight on a wire 58.6 centimeters long whose weight per centimeter is .0098 gram in order that it may make 256 complete vibrations per second?

Formula from physics. — The velocity v of a projectile at any moment is given by the formula

$$v = \sqrt{v_0^2 - 64\,y}$$

where v_0 is the initial velocity and y is the height of the projectile at this moment above the level of the starting point.

2. A projectile is fired from the ground with an initial velocity of 225 feet per second. How high will it be when it has a velocity of 150 feet per second?

3. Let P be a point on a semicircle whose diameter is AB, and let PR be the perpendicular from P to AB. Being given that $AB = 10$ inches, find the length of RP when $AR + RP = 7\frac{1}{2}$ inches. (RP is a mean proportional between AR and RB.)

4. Given the points P and Q with the coördinates $(\sqrt{3}, -2)$ and $(\sqrt{3}, 1)$ respectively. Find the point A on the y-axis such that $PA + AQ = 5$.

5. Find the coördinates of all the points on the line $y = \sqrt{2}$, the sum of whose distances from the points $(2, 0)$ and $(-2, 0)$ is equal to 5.

6. Find the coördinates of the points on the line $x = 3$ that are 10 units from the point $(9, 2)$.

7. A box is to be 3 feet long and 1 foot wide. How deep must it be in order to have a diagonal of 4 feet?

8. The points A and B on the x-axis have the abscissæ 4 and 12 respectively. What must be the abscissa of the point C on the x-axis in order that the area of the circle with C as a center and passing through A shall be twice that of the circle with C as a center and passing through B? Explain geometrically why there should be two answers.

9. What must be the abscissa of C in order that the area of the first circle of Problem 8 be one fourth that of the other one?

10. The points A and B on the x-axis have the abscissæ -3 and 8 respectively. What must be the radii of the two circles with centers at A and B respectively, and tangent to each other in order that the area of the former shall be five times that of the latter? Explain geometrically why there should be two answers.

11. The points A and B on the x-axis have the abscissæ a and b respectively. What must be the radii of two circles with centers at A and B respectively, and tangent to each other in order that the area of the former shall be k times that of the latter?

12. Find the abscissæ of the points in which the circle

$$(x-2)^2+(y+3)^2=25$$

cuts the x-axis. Draw the figure.

13. Find the abscissæ of the points in which the circle

$$(x+2)^2+(y+3)^2=9$$

cuts the x-axis. Draw the figure.

14. Where does the circle

$$(x+5)^2+(y-6)^2=16$$

cut the y-axis? Draw the figure.

15. What must be the dimensions of a rectangular field that is to contain 3 acres and have a perimeter of 100 rods?

16. What is the least possible perimeter of a rectangular field that is to contain 10 acres?

17. What is the least possible perimeter of a rectangular field that is to contain 20 acres?

18. What is the greatest possible area of a rectangular field inclosed by 200 rods of fence?

19. What is the area of the greatest rectangle that can be inclosed by a rope 60 feet long?

20. A man is in a boat 3 miles from the nearest point of a straight beach. If he rows at the rate of 4 miles an hour

and walks at the rate of 5 miles an hour, where ought he to land in order to reach a point on the beach 5 miles from the point directly opposite him in one hour and twenty-seven minutes?

Would it be possible for him to reach his destination in one hour?

21. An open box with a square base and a volume of 392 cubic inches is to be made by cutting 2 inch squares from the corners of a piece of tin and turning up the sides. How large a piece of tin should be used?

22. A man and a boy working together do a piece of work in 8 days. If we assume that the man could do it alone in 12 days less than the boy could do it, how long would it take each to do it?

23. The sides of two rectangles, one of which is within the other, are parallel and equal distances apart. How far apart are the sides if the outer rectangle is 10 centimeters by 14 centimeters and is twice as large as the inner one?

NOTE. — The equation obtained in the solution of a problem should express all the conditions of the problem if possible. But in some problems there are conditions that cannot be expressed in the equation. In Problem 23, for example, the implied condition that the number sought must be less than one half the smaller dimension of the outer rectangle cannot be expressed in the equation. Hence we must apply this condition to the roots after the equation has been solved and reject any that do not satisfy it.

24. Two men rowed 12 miles upstream and back in 7 hours and 30 minutes. The current was running at the rate of 3 miles an hour. How fast would they have gone in still water? Explain the two solutions.

25. A lever is to be cut from a bar weighing $3\frac{1}{2}$ pounds to the foot. How long must it be in order that it may balance about a point 2 feet from one end when a weight of 16 pounds is attached to this end? Explain the two solutions.

26. A stone is dropped over the edge of a cliff and 12 seconds later the sound of its striking the ground below is heard on the cliff. How high is the cliff if the velocity of sound is taken as 1120 feet per second?

27. Find the outer radius of a hollow spherical shell an inch thick whose volume is 125 cubic inches.

28. Find the outer edge of a hollow cubical shell an inch thick whose volume is 56 cubic inches.

29. The two legs of a right triangle are 3 inches and 4 inches long respectively. If the shorter one remain unchanged, by how much will the longer one need to be lengthened in order to lengthen the hypotenuse 2 inches?

30. If the longer leg of the right triangle of Ex. 29 remains unchanged, by how much will the shorter one need to be lengthened in order to lengthen the hypotenuse 2 inches?

Center of gravity. — Let A and B be two bodies of weights W_1 and W_2 respectively, and let the distances of their centers of gravity from the point 0 in a line with A and B be a and b respectively. Then, as explained in physics, the distance from 0 of the center of gravity of the two together is

$$\frac{W_1 a + W_2 b}{W_1 + W_2}.$$

31. A tin can open at the top has a diameter of 4 inches and a height of 6 inches. It weighs 4 ounces and its center of gravity is $2\frac{4}{7}$ inches above the center of its base. How much water must be poured into it in order to bring the center of gravity down to $2\frac{1}{2}$ inches. A cubic inch of water weighs $\frac{29}{50}$ of an ounce. Explain the two solutions.

CHAPTER VIII

SYSTEMS OF EQUATIONS IN TWO UNKNOWNS SOLVABLE BY MEANS OF QUADRATICS

82. Case I. *One of the equations is linear and the other one is a quadratic; that is, of the second degree.*

In this case we can proceed as illustrated in the following example.

Solve the equations
$$x^2 + y^2 = 16, \tag{1}$$
$$x + y = 1. \tag{2}$$

From (2) we get
$$y = 1 - x. \tag{3}$$

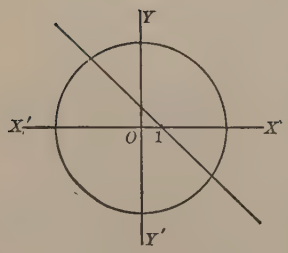
Substituting this value of y for y in (1), we get

$$x^2 + (1 - x)^2 = 16,$$
$$2x^2 - 2x - 15 = 0,$$
$$x = \frac{1 \pm \sqrt{31}}{2}.$$

Then from (3)
$$y = \frac{1 \mp \sqrt{31}}{2}.$$

Hence the pairs of values of x and y that satisfy both equations are

$$x = \frac{1 + \sqrt{31}}{2}, \qquad x = \frac{1 - \sqrt{31}}{2},$$
$$y = \frac{1 - \sqrt{31}}{2}, \qquad y = \frac{1 + \sqrt{31}}{2}.$$



The student is familiar with the fact that any linear equation in x and y has for its locus a straight line. He also knows that the locus of equation (1) is a circle with its center at the origin and radius equal to 4. Since the pairs of values of x and y just found satisfy both equa-

tions, they are the coördinates of the points of intersection of this circle and the straight line represented by equation (2).

If only one pair of values of x and y satisfy both equations, the straight line is tangent to the curve. If the values of x and y that satisfy both equations are imaginary, the line does not meet the curve. Conversely, if the line does not meet the curve, we know that the values of x and y which satisfy both equations are imaginary, but we do not know from this fact their particular values.

If the equations are satisfied by real values of x and y, these values can be determined graphically by drawing the locus of each equation and measuring the coördinates of their common points.

EXERCISES

Solve each of the following pairs of equations. Check your work by means of graphs. The loci of the first equations in Exs. 11 and 12 are circles whose centers are not at the origin. Do not attempt to draw these loci.

1. $x^2 + y^2 = 9,$
   $x + y = 3.$

2. $x^2 + y^2 = 4,$
   $2x + y = 5.$

3. $x^2 + y^2 = 16,$
   $3x - 2y = 0.$

4. $x^2 + y^2 = 1,$
   $3x + 4y + 1 = 0.$

5. $x^2 + y^2 = 10,$
   $y = x - 1.$

6. $x^2 + y^2 = 25,$
   $2x + 4y = 3.$

7. $x^2 + y^2 = 40,$
   $x + 4y + 5 = 0.$

8. $x + y = 0,$
   $x^2 + y^2 = 35.$

9. $2x = 5y,$
   $x^2 + y^2 = 12.$

10. $x^2 = 10 - y^2,$
    $y = x - 10.$

11. $x^2 + y^2 - 4x - 6y + 9 = 0,$
    $y = x + 2.$

12. $x^2 + y^2 - 6x + 5y = 2,$
    $3x + 2y = 1.$

13. $y^2 = 6 - x^2$,
$y = 3x + 1$.

14. $x^2 - 9 = -y^2$,
$2x + y + 1 = 0$.

15. $x^2 + y^2 = 1$,
$x + y = 1$.

16. $x^2 + y^2 - 9 = 0$,
$y = 2$.

17. $x^2 + y^2 - 9 = 0$,
$y = -3$.

18. $x^2 + y^2 = 9$,
$x = 4$.

19. $y - 1 = 3(x + 1)$,
$x^2 + y^2 = 3$.

20. $y + 2 = 4(x + 3)$,
$x^2 + y^2 = 16$.

83. Equation of an ellipse. — Not all equations of the second degree in x and y represent circles. We have already met with equations of this kind that represent parabolas (see § 72), and there are equations of this kind that represent other curves.

Consider, for example, the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

The student should verify that the following pairs of values of x and y satisfy this equation:

x	0	0	3	−3	1	1	2	2	−1	−1	−2	−2
y	2	−2	0	0	$\frac{4}{3}\sqrt{2}$	$-\frac{4}{3}\sqrt{2}$	$\frac{2}{3}\sqrt{5}$	$-\frac{2}{3}\sqrt{5}$	$\frac{4}{3}\sqrt{2}$	$-\frac{4}{3}\sqrt{2}$	$\frac{2}{3}\sqrt{5}$	$-\frac{2}{3}\sqrt{5}$

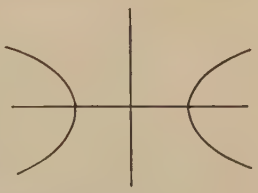
Moreover, $x = \pm\frac{3}{2}\sqrt{4 - y^2}$ and $y = \pm\frac{2}{3}\sqrt{9 - x^2}$. Hence if the absolute value of y exceeds 2, x is imaginary; and if the absolute value of x exceeds 3, y is imaginary. This means that there is no point on the locus whose ordinate exceeds 2 in absolute value and no point whose abscissa exceeds 3 in absolute value.

If we plot the points whose coördinates are the pairs of values of x and y in the foregoing table and connect these points by a smooth curve, we shall have the graph of the equation.

This curve is called an **ellipse**.

84. Equation of a hyperbola. — We consider next the equation

$$\frac{x^2}{9}-\frac{y^2}{4}=1.$$

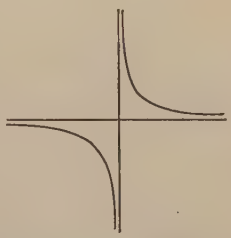


Solving for y, we get $y=\pm\frac{2}{3}\sqrt{x^2-9}$. Hence there is no point on the locus whose abscissa is less than 3 in absolute value. Solving for x, we get $x=\pm\frac{3}{2}\sqrt{y^2+4}$. Hence there is no limitation on the ordinates of the points of the locus.

A table of corresponding values of x and y is as follows:

x	3	-3	4	4	-4	-4	5	5	-5	-5	6	6	-6	-6
y	0	0	$\frac{2}{3}\sqrt{7}$	$-\frac{2}{3}\sqrt{7}$	$\frac{2}{3}\sqrt{7}$	$-\frac{2}{3}\sqrt{7}$	$\frac{8}{3}$	$-\frac{8}{3}$	$\frac{8}{3}$	$-\frac{8}{3}$	$2\sqrt{5}$	$-2\sqrt{5}$	$2\sqrt{5}$	$-2\sqrt{5}$

This curve is called a **hyperbola.**



Curves whose equations are of the form $xy=k$ are also hyperbolas, but they are situated differently with respect to the coördinate axes. The figure gives the graph of the equation

$$xy=1.$$

When k is positive, the two branches of the curve are in the first and third quadrants; when k is negative, the two branches are in the second and fourth quadrants.

85. Equation of a parabola. — The graph of the equation

$$y^2=8x$$

is given in the figure. The curve is a **parabola,** but situated differently with respect to the coördinate axes from the parabola mentioned in § 72.

86. Equation of two straight lines. — If the equation is such that when the right member is zero the left member can be factored into two factors of the first degree, the locus is composed of two straight lines.

Consider, for example, the equation

$$3x^2 + 5xy + 11x + y + 6 = 2y^2.$$

Transposing and factoring, we get

$$(x + 2y + 3)(3x - y + 2) = 0.$$

Now any pair of values of x and y that make either of these factors zero must satisfy the equation and are therefore the coördinates of a point on the locus. And conversely the coördinates of any point on the locus when substituted for x and y respectively must make the product of these factors zero (10, Chap. I). Hence the locus of the given equation is the same as the loci of the two equations

$$x + 2y + 3 = 0,$$
$$3x - y + 2 = 0.$$

87. These are the simpler forms of the equations of the second degree in two variables. But there are many others. The equations of circles, ellipses, hyperbolas, parabolas, and of two straight lines are of the second degree.

EXERCISES

Solve each of the following pairs of simultaneous equations. Check your work by means of graphs.

1. $\frac{x^2}{9} + \frac{y^2}{4} = 1,$
   $y = 3x + 1.$

2. $xy = -2,$
   $2x + y + 1 = 0.$

3. $\frac{x^2}{9} - \frac{y^2}{4} = 1,$
   $y + 3 = \frac{1}{2}(x + 1).$

4. $y^2 = 6x,$
   $y - 3 = -2(x - 1).$

5. $y = 4x + 1,$
   $y^2 - 8x = 0.$

6. $xy = 1,$
   $4x - 2y + 3 = 0.$

7. $x^2 + 6xy + 9y^2 = 4,$
   $\frac{x}{3} + \frac{y}{4} = 1.$

8. $\frac{x^2}{9} + \frac{y^2}{16} = 1,$
   $\frac{x}{4} - \frac{y}{5} = 1.$

9. $x^2 - \frac{y^2}{16} = 1,$
$3x + 2y = 0.$

10. $y^2 + x = 0,$
$2y + x = 0.$

11. $3xy + 4 = 0,$
$\frac{x}{5} + \frac{y}{3} + 1 = 0.$

12. $4y^2 - 5x = 0,$
$y - 2 = 3(x + 1).$

13. $4x^2 + 5xy + x = 0,$
$2x - y + 4 = 0.$

14. $\frac{x^2}{3} + \frac{y^2}{2} = 1,$
$x + y + 1 = 0.$

15. $\frac{x^2}{4} - \frac{y^2}{25} = 1,$
$2x + 5y + 1 = 0.$

16. $\frac{x^2}{4} + \frac{y^2}{5} = 1,$
$\frac{x}{9} - \frac{y}{16} = 1.$

17. $x^2 - y^2 = 0,$
$3x - 2y = 4.$

18. $xy - 5 = 0,$
$y = 2x - 1.$

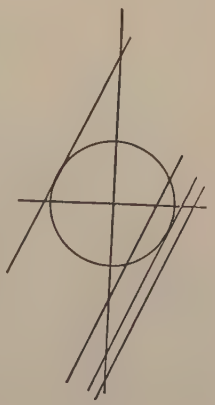
19. $7x^2 - 3xy + 2x = 0,$
$5x + y + 2 = 0.$

20. $\frac{x^2}{9} - \frac{y^2}{4} = 0,$
$x + y + 1 = 0.$

21. $x^2 - 5y = 0,$
$x - y = 1.$

88. It is evident that the intersections of the line $y = 3x + b$ and the circle $x^2 + y^2 = 10$ depend upon the value of b, since this value determines the position of the locus of the equation

$$y = 3x + b.$$



If we eliminate y between the two equations, we get

$$x^2 + (3x + b)^2 = 10,$$

$$10x^2 + 6bx + b^2 - 10 = 0.$$

If now b has a value that makes

$$36b^2 - 40(b^2 - 10) > 0,$$

the roots of this equation are real and distinct and therefore the line cuts the circle in two points. But if b has a value that makes

$$36\,b^2 - 40(b^2 - 10) = 0,$$

this equation has only one root and the line is tangent to the circle. If finally b has a value that makes

$$36\,b^2 - 40(b^2 - 10) < 0,$$

this equation has imaginary roots and the line has no points in common with the circle.

Now $\quad 36\,b^2 - 40(b^2 - 10) = -4\,b^2 + 400.$

If then $\quad -4\,b^2 + 400 > 0,$

or $\quad b^2 < 100,$ (See § 182.)

the line cuts the circle. But in order that b^2 be less than 100, b must have a value between -10 and 10; that is, $-10 < b < 10$. If

$$-4\,b^2 + 400 = 0,$$

$$b = \pm 10,$$

and if $\quad -4\,b^2 + 400 < 0,$

then $\quad b^2 > 100,$

and either $\quad b < -10$ or $b > 10.$

The foregoing results are summarized in the following table:

$-10 < b < 10$	Line cuts circle in two points.
$b = -10$, or $b = 10$	Line is tangent to circle.
$b < -10$, or $b > 10$	Line has no point in common with circle.

Definition. — A literal constant that occurs in the equation of a curve is called a **parameter.**

Thus in the equation $y = 3x + b$, b is a parameter.

If the equation of a straight line contains a parameter, we can frequently determine, as in the example just given, the values of the parameter for which the line cuts the locus of a given equation of the second degree in two points; the values of the parameter for which the line has only one point in common with this locus; and the values of the parameter for which the line has no points in common with this locus.

EXERCISES

Solve each of the following pairs of equations. Find the values of the parameter for which the line has but one point in common with the curve. Find also at least one value of the parameter for which the line cuts the curve in two points, and at least one value for which the line and the curve have no common point.

1. $x^2+y^2=16$, $y=mx+5$.
2. $x^2+y^2=16$, $y=2x+b$.
3. $x^2+y^2=1$, $\frac{x}{2}+\frac{y}{m}=1$.
4. $x^2+y^2=9$, $y-4=m(x-5)$.
5. $\frac{x^2}{4}+y^2=1$, $y=3x+b$.
6. $\frac{x^2}{4}-y^2=1$, $y=3x+b$.
7. $\frac{x^2}{9}-\frac{y^2}{16}=1$, $x+y=a$.
8. $x^2+y^2=49$, $y=mx+7$.
9. $\frac{x^2}{9}+\frac{y^2}{16}=1$, $y=b$.
10. $xy=1$, $y=2x+b$.
11. $-x^2+y^2=16$, $y=2x+b$.
12. $xy=1$, $x+y=k$.
13. $y^2=3x$, $y=mx+1$.
14. $xy+4=0$, $y=mx$.
15. $\frac{x^2}{4}+\frac{y^2}{9}=0$, $y=mx+3$.

89. Case II. *Each equation contains each unknown only to the second power; that is, there are no terms of the first degree and no terms containing xy.*

In this case we first solve the equations for x^2 and y^2 and then from these values get x and y. The general procedure is illustrated in the following example:

Solve the equations
$$x^2+y^2=16, \tag{1}$$
$$\frac{x^2}{25}+\frac{y^2}{9}=1. \tag{2}$$

Eliminating y^2, we get
$$16x^2=175,$$
$$x=\frac{\pm 5\sqrt{7}}{4}.$$

Eliminating x and solving for y, we get

$$y = \pm \tfrac{9}{4}.$$

By substituting these values of x and y for x and y respectively in equations (1) and (2), we see that either value of x can go with either value of y.

Thus,

$$x = \frac{5\sqrt{7}}{4},\ \frac{5\sqrt{7}}{4},\ \frac{-5\sqrt{7}}{4},\ \frac{-5\sqrt{7}}{4};$$

$$y = \frac{9}{4},\ -\frac{9}{4},\ \frac{9}{4},\ -\frac{9}{4}.$$

Hence there are four solutions to these equations and the loci intersect in four points.

EXERCISES

Solve each of the following pairs of equations. Check your work by means of graphs. Draw the graphs of the equations in Exs. 1–3 with reference to the same set of axes; also those in Exs. 4–7; and those in Exs. 8, 9.

1. $x^2 + y^2 = 16,$
   $\frac{x^2}{9} - \frac{y^2}{25} = 1.$

2. $x^2 + y^2 = 9,$
   $\frac{x^2}{9} - \frac{y^2}{25} = 1.$

3. $x^2 + y^2 = 1,$
   $\frac{x^2}{9} - \frac{y^2}{25} = 1.$

4. $\frac{x^2}{25} + \frac{y^2}{9} = 1,$
   $x^2 + y^2 = 25.$

5. $x^2 + y^2 = 9,$
   $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

6. $x^2 + y^2 = 4,$
   $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

7. $x^2 + y^2 = 36,$
   $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

8. $\frac{x^2}{25} + \frac{y^2}{9} = 1,$
   $\frac{x^2}{25} - \frac{y^2}{9} = 1.$

9. $\frac{x^2}{16} + \frac{y^2}{9} = 1,$
   $\frac{x^2}{25} - \frac{y^2}{9} = 1.$

10. $\frac{x^2}{16} + \frac{y^2}{9} = 1,$
    $\frac{x^2}{25} - \frac{y^2}{9} = 0.$

11. $x^2 + y^2 = 9,$
    $x^2 = 4 - y^2.$

12. $x^2 + y^2 = 25.$
    $2x^2 + 3y^2 = 60.$

13. $\frac{x^2}{5} + \frac{y^2}{4} = 1,$
    $\frac{x^2}{4} + \frac{y^2}{5} = 1.$

14. $\frac{x^2}{9} - \frac{y^2}{4} = 1,$
    $\frac{x^2}{4} + y^2 = 1.$

15. $x^2 - y^2 = 0$, $x^2 + y^2 = 8$.

16. $x^2 + y^2 = 25$, $\frac{x^2}{5} - \frac{y^2}{4} = 1$.

17. $x^2 + y^2 = 25$, $\frac{x^2}{5} + \frac{y^2}{4} = 1$.

18. $3x^2 + 5y^2 = 32$, $x^2 = y^2$.

19. $4x^2 = 9y^2$, $4x^2 + 9y^2 = 1$.

20. $x^2 + y^2 = 1$, $x^2 - y^2 = 1$.

90. We have discussed in some detail the two simplest cases that can arise in the solution of sets of equations in two unknowns solvable by quadratics. In general, the solution of a set of two equations in two unknowns is beyond the scope of this book. There are, however, certain ingenious special devices that can sometimes be used. We shall explain two of these. The others are not sufficiently important to the student to justify a discussion of them here.

(*a*) Solve the equations
$$x^3 + y^3 = 27, \tag{1}$$
$$x + y = 3. \tag{2}$$

Equation (1) can be written
$$(x + y)(x^2 - xy + y^2) = 27. \tag{3}$$

If, in the left member of (3) we substitute for $x + y$ its value as given by (2), we get
$$3(x^2 + xy + y^2) = 27,$$
or
$$x^2 + xy + y^2 = 9, \tag{4}$$

Now (2) and (4) can be solved in the manner explained under Case I. The graph of equation (1) is too complicated to be given here.

(*b*) Solve the equations
$$x^2 + y^2 = 36, \tag{1}$$
$$xy = 14. \tag{2}$$

If we multiply each member of (2) by 2 and add the members of the resulting equation to the corresponding members of (1), we get
$$x^2 + 2xy + y^2 = 64. \tag{3}$$
Hence
$$x + y = 8, \tag{4}$$
or
$$x + y = -8. \tag{5}$$

We shall get all the solutions of (1) and (2) by solving (2) and (4) and also (2) and (5).

EXERCISES

Solve each of the following pairs of equations. When neither of the equations is of third degree, check your results by means of graphs. When one of the equations is of the third degree, check your results by direct substitution in both the original equations.

1. $x^2+y^2=10$, $xy=3$.
2. $x^3-y^3=8$, $x-y=2$.
3. $8x^3+27y^3=1$, $2x+3y=1$.
4. $x^2+y^2=20$, $xy=8$.
5. $4x^2+9y^2=1$, $xy=2$.
6. $x^2+16y^2=44$, $xy+1=0$.
7. $x^3+y^3=64$, $x+y=4$.
8. $x^3-y^3=1$, $x-y=1$.
9. $x^2+y^2=16$, $xy=10$.
10. $x^3+y^3=1$, $x+y=1$.
11. $x^3-y^3=2$, $x-y=1$.
12. $16x^2+25y^2=9$, $xy=1$.
13. $64x^3+y^3=4$, $4x+y=2$.
14. $x^2+y^2=5$, $xy=2$.
15. $xy+4=0$, $x^2+9y^2=25$.

PROBLEMS

1. The sum of the squares of two numbers is 193 and the product of the numbers is 84. What are the numbers?

2. The sum of the squares of two numbers is 250 and the sum of the numbers is 22. What are the numbers?

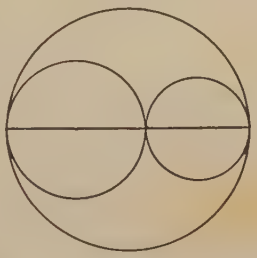
3. The difference of the cubes of two numbers is 819, and the difference of the numbers is 3. What are the numbers?

4. It is known that the sum of the lengths of two cubical bins is 29 feet and that their combined capacity is 6119 cubic feet. What are the dimensions of each bin?

5. The area of a right triangle is 30 square inches and its hypotenuse is 13 inches. Find the lengths of the legs.

6. The diagonal of a rectangular field containing $11\frac{1}{4}$ acres is 80 rods. What are the dimensions of the field?

7. Two circles with their centers on the same diameter of a third circle are tangent to each other externally and to the third circle internally, as in the figure. What must be the radii of the two inner circles in order that the sum of their areas be one half the area of the outer circle, the radius of the latter being 10 inches?



8. What must be the radii of the inner circles of Problem 7 in order that the sum of their areas be three fourths that of the outer circle? How will the sum of the circumferences of the inner circles compare with the circumference of the outer circle?

9. What is the least possible combined area of two inner circles related to a circle of radius 10 inches in the way described in Problem 7?

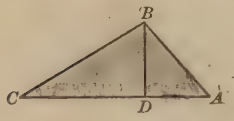
HINT. — Let this combined area be πa, and then find the least value of a for real values of the radii of the inner circles.

10. Two spheres with their centers on the same diameter of a third sphere are tangent externally to each other and internally to the third sphere. What must be the radii of the two inner spheres in order that the sum of their volumes be three fourths the volume of the outer sphere, the radius of the latter being 10 inches?

11. What is the least possible total volume of two inner spheres related to a sphere of radius 10 inches in the way described in Problem 10?

12. What is the least possible total surface of the inner spheres of Problem 11?

13. Being given that the sides AB, BC, and CA of the triangle ABC are 8 inches, 12 inches, and 15 inches respectively, find the altitude BD.



Let $BD=x$ and $AD=y$. Then $DC=15-y$, and we have the two equations

$$x^2 + y^2 = 64,$$
$$x^2 + (15 - y)^2 = 144.$$

These equations do not come under any of the cases we have discussed. The method of solution is, however, obvious.

When one altitude is known, the others can easily be found from the fact that the product of any side of a triangle and the altitude upon that side equals twice the area of the triangle.

14. Find the three altitudes of the triangle whose sides are 20, 24, and 30 respectively.

15. Find one altitude and the area of the triangle whose sides are 7 centimeters, 10 centimeters, and 15 centimeters respectively.

CHAPTER IX

PROGRESSIONS

91. Arithmetic progression. — A succession of numbers so related that each one is obtained by adding a fixed amount to the preceding one is called an **arithmetic progression.**

The numbers forming the progression are called its **terms.** The fixed amount which must be added to any term to get the following one is called the **common difference.**

92. If a_1 is the first term of the progression and d the common difference, the progression is

$$a_1,\ a_1+d,\ a_1+2\,d,\ a_1+3\,d,\ \cdots$$

The succession of dots after $a+3\,d$ means "and so forth."

For example, 3, 7, 11, 15, 19, 23, 27;
and 5, 3, 1, -1, -3, -5, -7, -9, -11

are arithmetic progressions. In the former the common difference is 4 and in the latter it is -2.

If a_n denotes the nth term of an arithmetic progression, it is clear that

$$a_n = a_1+(n-1)d. \qquad \text{(I)}$$

This formula involves the four numbers a_1, d, n, and a_n; and by means of it we can find any one of them when the other three are given.

93. The sum of n consecutive terms of an arithmetic progression. — If we denote by s_n the sum of the first n terms of an arithmetic progression, then

$$s_n = a_1+[a_1+d]+[a_1+2\,d]+\cdots+[a_1+(n-3)d] \\ +[a_1+(n-2)d]+[a_1+(n-1)d].$$

The symbol formed by the three dots is to be read "and so on up to."

If we write the right member of this equation in the reverse order, we have

$$s_n = [a_1 + (n-1)d] + [a_1 + (n-2)d] + [a_1 + (n-3)d] + \cdots + [a_1 + 2d] + [a_1 + d] + a_1.$$

Adding the corresponding members of these two equations, we get

$$2s_n = [2a_1 + (n-1)d] + [2a_1 + (n-2)d + d] + [2a_1 + (n-3)d + 2d] + \cdots + [2a_1 + (n-3)d + 2d] + [2a_1 + (n-2)d + d] + [2a_1 + (n-1)d].$$

Now there are n brackets in the right member of this equation and each of them is equal to $2a_1 + (n-1)d$. Hence

$$2s_n = n[2a_1 + (n-1)d],$$

and

$$s_n = \frac{n}{2}[2a_1 + (n-1)d].$$

If we take into consideration the value of a_n as given by Formula I, we see that

$$s_n = \frac{n}{2}(a_1 + a_n). \qquad \text{(II)}$$

Formulæ I and II are of great importance in problems involving arithmetic progressions. They involve five numbers a_1, d, n, a_n, and s_n. If any three of these are given, the other two can be found by means of these formulæ.

94. Series. — An algebraic expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n$$

is called a **series.** The a's are called the **terms** of the series.

When we speak of the sum of a series we mean the sum indicated by the series. Thus, Formula II gives us the sum of an arithmetic series.

95. Arithmetic means. — The terms of an arithmetic progression between the first one and the last one are called **arithmetic means.**

By the aid of Formula I any number of arithmetic means can be inserted between two given numbers; that is, an arith-

metic progression can be formed with the two given numbers as the first term and the last term respectively and with the given number of intermediate terms.

Example. — Insert 7 arithmetic means between 2 and 26.

Here the first term of the progression to be formed is 2 and the ninth one is 26. Besides these two terms there are to be 7 intermediate terms.

Hence, $a_1 = 2,\ n = 9,\ a_9 = 26.$

Substituting these values in Formula I, we get

$$26 = 2 + (9 - 1)d.$$

Hence, $d = 3,$

and the required means are

5, 8, 11, 14, 17, 20, 23.

The student is doubtless already familiar with the problem of inserting one arithmetic mean between two numbers, since this is the same as the problem of finding the average of two numbers. If M is the average of a_1 and a_3, then

$$M = \frac{a_1 + a_3}{2},$$

and a_1, M, and a_3 form an arithmetic progression. For this reason M is called **the arithmetic mean** of a_1 and a_3.

EXERCISES AND PROBLEMS

Determine which of the four following sets of numbers form arithmetic progressions:

1. 7, 10, 13, 15.
2. -4, 0, 4, 8, 12.
3. 20, 13, 6, -1, -8.
4. 1, -1, -2.

5. Solve Formula I for a_1, n, and d in turn.

6. What is the last term of an arithmetic progression of 12 terms if the first term is -2 and the common difference is 2?

7. There are 20 terms in an arithmetic progression of which the last one is -45, and the common difference is $4\frac{1}{4}$. What is the first term?

8. What is the common difference when 27 is the first of twelve terms of an arithmetic progression and -13 is the last?

9. How many terms are there in an arithmetic progression of which the first term is 321, the last term 0, and the common difference -3?

10. Insert four arithmetic means between 7 and 20.

11. What is the arithmetic mean of 20 and -20?

12. Insert twelve arithmetic means between 14 and -15, and find the sum of the terms of the resulting progression.

13. Given $a_1 = \frac{1}{4}$, $s_{12} = 102$; find d and a_{12}.

14. If a clock strikes the hours, how many strokes a day does it make?

15. Find the sum of the first fifty odd numbers.

16. Prove that the sum of the first n odd numbers is n^2.

17. A freely falling body falls approximately 16 feet the first second and in each second thereafter 32 feet more than in the preceding second. A stone dropped from the top of a tower strikes the ground in 4 seconds. How high is the tower and how far did the stone fall the last second?

18. Find the sum of all the integers between 100 and 200 that are divisible by 3.

19. A ball rolling down an incline of 30° goes 8 feet the first second and in each second thereafter 16 feet more than in the preceding second. How far will it roll in 9 seconds?

20. Show that the distances of the following points from the origin are in arithmetic progression:

$$(1, 1\tfrac{1}{3}),\ (2, 2\tfrac{2}{3}),\ (3, 4),\ (4, 5\tfrac{1}{3}),\ (5, 6\tfrac{2}{3}),\ \text{and}\ (6, 8).$$

Plot these points and see if they lie on a straight line.

96. Geometric progression. — A succession of numbers so related that the ratio of each one to the preceding one is a fixed number is called a **geometric progression.**

Thus the numbers 3, 6, 12, 24, 48 and 96 form a geometric progression.

The numbers forming a progression are called its **terms.** The ratio of any term to the preceding one is called the **common ratio** of the progression, and is usually represented by the letter r. The first term, the nth term, the number of terms, and the sum of the terms are represented by the same letters as the corresponding numbers in an arithmetic progression. We shall assume throughout that $r \neq 1$.

97. The terms of a geometric progression can be represented in the following way:

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ \cdots.$$

From this it is obvious that the nth term of the progression is ar^{n-1}. That is, in a geometric progression of n terms

$$a_n = a_1r^{n-1}. \qquad \text{(I)}$$

This formula involves the four numbers, a_1, a_n, n, and r; and by means of it we can find any one of them when the other three are given.

98. The sum of n consecutive terms of a geometric progression. —

$$s_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}.$$

Multiplying each member of this equation by r, we get

$$s_nr = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n.$$

Hence, by subtraction,

$$s_n - s_nr = a_1 - a_1r^n.$$

All the terms, except two, in the right members of these equations are destroyed by the subtraction.

Solving the last equation for s_n, we get

$$s_n = \frac{a_1(1 - r^n)}{1 - r}. \qquad \text{(II)}$$

The student can verify that Formula II is correct by dividing the numerator of the right member by the denominator.

If we take Formula I into consideration, we can write Formula II in the following form:

$$s_n = \frac{a_1 - a_n r}{1 - r}. \qquad \text{(III)}$$

By means of Formulae I and II or I and III we can find any two of the five numbers a_1, a_n, n, r, and s_n when the other three are given.

99. Geometric means. — The terms of a geometric progression between the first one and the last one are called **geometric means.**

In a geometric progression of three terms the middle term is called **a geometric mean** of the other two.

By the aid of Formula I any number of geometric means can be inserted between two given numbers.

Example. — Insert four geometric means between 5 and 160.

In the formula

$$a_n = a_1 r^{n-1}$$

$a_1 = 5$, $n = 6$, and $a_6 = 160$ since there are to be four terms besides the first one and the last one. Hence

$$5r^5 = 160,$$
$$r^5 = 32,$$
$$r = 2,$$

and the required geometric means are 10, 20, 40, and 80.

In discussing arithmetic progressions we found only one way to insert the required number of arithmetic means be-

tween two given numbers. But the required number of geometric means can be inserted between two given numbers in more than one way.

Suppose, for example, we wish to insert three geometric means between 6 and 1536. Here $a_1 = 6$, $n = 5$, and $a_5 = 1536$. Hence

$$6r^4 = 1536,$$
$$r^4 = 256,$$
$$r = \pm 4.$$

The required means are therefore either

24, 96, and 384;

or -24, 96, and -384.

In this example the common ratio can be any number whose fourth power is 256. Now it will be shown in § 138 that there are four such numbers. However, only two of them are real, and we are considering here only real numbers. This explains why in the first example we gave only one set of geometric means. As a matter of fact, there are five such sets, since there are five numbers whose fifth power is 32.

The problem of inserting one geometric mean between two numbers is the same as that of finding a mean proportional between these numbers.

Two real numbers of opposite signs have no real geometric mean, since the square of such a mean would necessarily be negative.

EXERCISES AND PROBLEMS

Determine which of the four following sets of numbers form geometric progressions:

1. $-2, 4, -8, 16.$
2. 5, 7, 10, 13.
3. $-2, -4, -8, -16.$
4. 1, 4, 16, 32.

5. Solve Formula I for a_1 and r in turn.

6. Find the tenth term of a geometric progression whose first three terms are 4, $\frac{4}{9}$, and $\frac{4}{81}$ respectively.

7. The first and fourth terms of a geometric progression are 2 and 54 respectively. Find the common ratio and the second term.

8. What is the fifth term of a geometric progression whose first two terms are a and b respectively?

9. The first term of a geometric progression is 7 and the common ratio is 3. What is the sum of the first eight terms?

10. Insert three geometric means between 6 and 486.

11. Which is the greater, the arithmetic mean of 4 and 16, or the positive geometric mean of these numbers?

12. Write both the arithmetic mean and the positive geometric mean of the positive numbers a and b. Then select four sets of values for a and b and determine which of these means is the greater for these values of a and b.

13. What will \$100 amount to in five years at 3%, compounded annually?

14. What will p dollars amount to in n years at r per cent, compounded annually? compounded quarterly?

15. Solve Formula III for each letter in terms of the others.

16. Show that the distances from the origin of the points $(\frac{4}{3}, 1)$, $(\frac{8}{3}, 2)$, $(1\frac{6}{3}, 4)$, $(\frac{32}{3}, 8)$, and $(\frac{64}{3}, 16)$ form a geometric progression and find their sum.

17. Each stroke of an air pump exhausts one fifteenth of the air in the receiver. How much of the air originally in the receiver is removed in ten strokes?

18. Given $a_1 = 3$, $r = 2$; find a_9 and s_9.

19. The perpendicular distance from the center of a circle to a side of an inscribed equilateral triangle is one half the radius of the circle. An equilateral triangle is inscribed in a circle whose radius is 6 inches. Then a circle is inscribed in the triangle and an equilateral triangle in this circle and so on until there are five circles in all. What is the sum of their circumferences? What is the sum of the perimeters of the triangles?

20. The second of a series of squares has its vertices at the mid-points of the sides of the first one, and, in general, each square of the series has its vertices at the mid-points of the sides of the preceding one. What are the area and perimeter of the tenth square, if each side of the first square is 2 inches?

100. Limit of certain geometric series. — Consider the geometric progression whose first term is 1 and whose common ratio is $\frac{1}{2}$. The sum of the first n terms is given by the formula

$$s_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - (\tfrac{1}{2})^{n-1}.$$

It will be observed that this sum is always less than 2, no matter how many terms of the progression we take; but that it differs from 2 by an amount that decreases as the number of terms taken increases. By adding together enough terms of this series we can get a sum that differs from 2 by as small an amount as we please, and the difference will be still less if we take more terms. Thus the sum of ten terms lacks $\frac{1}{512}$ of being equal to 2, and the sum of twenty terms lacks $\frac{1}{524288}$ of being equal to 2.

We express these facts concisely by saying that 2 is the **limit** of the sum of n terms of this progression as n is increased without limit.

In general, we have

$$s_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} r^n.$$

As we give to n larger and larger values, the fraction $\dfrac{a}{1 - r}$ does not change, while if the absolute value of r is less than 1, $\dfrac{a}{1 - r} \cdot r^n$ gets nearer and nearer to zero, and we can make it as near to zero as we wish by making n large enough.* This is

* Not all variables that are getting nearer and nearer to zero can get as near to zero as we wish. A variable might, for example, be constantly getting smaller and yet never get as small as one.

due to the fact that by taking n large enough we can make r^n as near to zero as we wish. Hence s_n can be made to differ from $\frac{a}{1-r}$ by as little as we please by making n large enough. We express this by saying that $\frac{a}{1-r}$ is the *limit* of s_n as n is increased without limit, or that $\frac{a}{1-r}$ is the limit of the sum of the geometric series whose common ratio is r, as the number of terms is increased without limit. (See § 202.)

When the absolute value of r is greater than 1, the fraction $\frac{a}{1-r} \cdot r^n$ does not get nearer to zero as n increases. Hence the preceding argument does not apply. Such infinite series are not considered here. (See § 203.)

The repeating decimal $.666\cdots$

can be written thus $.6 + .06 + .006 + \cdots$.

It is therefore an infinite geometric series whose first term is .6 and whose common ratio is .1. Hence

$$s_n = \frac{.6}{1-.1} - \frac{.6}{1-.1}(.1)^n.$$

This approaches the limit $\frac{.6}{1-.1}$ or $\frac{2}{3}$ when n increases without limit.

EXERCISES

Find the limit of the following geometric series as the number of terms increases without limit.

1. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$.
2. $1 - \frac{1}{3} + \frac{1}{9} - \cdots$.
3. $5 + \frac{15}{4} + \frac{45}{16} + \cdots$.
4. $6 - 4 + \frac{8}{3} - \cdots$.
5. $\frac{3}{4} - \frac{9}{16} + \frac{27}{64} - \cdots$.
6. $4 + \frac{8}{3} + \frac{16}{9} + \cdots$.
7. $2 + \frac{1}{2} + \frac{1}{2} + \cdots$.
8. $2 - \frac{1}{2} + \frac{1}{8} - \cdots$.

Find the limiting value of each of the following repeating decimals:

9. .333 ....
10. .07777777 ....
11. 1.66666 ....
12. 1.111 ....
13. 20.202020 ....
14. 2.555 ....
15. 3.544242 ....
16. .0065334334 ....
17. 1.21212 ....
18. 32.34125125 ....
19. .00041616 ....
20. 5.2330404 ....

101. Harmonic progression. — A succession of numbers so related that their reciprocals form an arithmetic progression is called a **harmonic progression.**

Thus the numbers

$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{3}, \quad \tfrac{1}{4}, \quad \tfrac{1}{5}, \quad \text{etc.}$$

form a harmonic progression, and it is from this particular progression that the name "harmonic" is derived. If a set of stretched wires or strings whose lengths are proportional to the terms of this progression are set vibrating, the sound produced is harmonious.

We shall have no occasion in what follows to use harmonic progressions.

Ex. What can you say of a succession of numbers so related that their reciprocals form a geometric progression?

CHAPTER X

PERMUTATIONS AND COMBINATIONS

102. Permutations. — In certain problems it is necessary to find out in how many orders a given number of things can be arranged. This information is obtained by counting. But it is not necessary actually to make this count in every problem that presents itself. The result obtained in a typical problem can be embodied in a formula, and then, in order to solve any problem of this type, we have only to substitute the proper numbers in this formula.

Each arrangement of a number of things in a definite order is called a permutation of these things.

Suppose we have three things represented by the letters a, b, c respectively. They can be arranged in the six following orders:

$$\begin{array}{ccc} a\ b\ c & b\ a\ c & c\ a\ b \\ a\ c\ b & b\ c\ a & c\ b\ a \end{array}$$

That is, six different permutations of three things are possible.

Sometimes when we are considering n things and r is some positive integer less than n, we wish to find the total number of possible permutations of these n things when in each permutation we use only r things, or, as we shall say, the number of permutations of n things taken r at a time. Thus, all the possible permutations of four things taken three at a time are given in the following table:

$$\begin{array}{cccc} a\ b\ c & a\ b\ d & a\ c\ d & b\ c\ d \\ a\ c\ b & a\ d\ b & a\ d\ c & b\ d\ c \\ b\ a\ c & b\ a\ d & c\ a\ d & c\ b\ d \\ b\ c\ a & b\ d\ a & c\ d\ a & c\ d\ b \\ c\ a\ b & d\ b\ a & d\ c\ a & d\ c\ b \\ c\ b\ a & d\ a\ b & d\ a\ c & d\ b\ c \end{array}$$

It is easy to see by counting that there are 24 of these.

In order to derive a general formula for the number of permutations of n things taken r at a time, r being a positive integer less than, or equal to, n, we shall consider first a preliminary theorem.

103. Illustrative example. — If there are two roads from A to B and three from B to C, by how many routes can one go from A through B to C?

Either of the roads from A to B can be followed by any one of the three from B to C. Hence there are $3 \cdot 2$, or 6, routes by which one can go from A through B to C.

This suggests the following general

Theorem. — *If one act can be performed in p ways; and if, after this has been performed in one of these ways, a second act can be performed in q ways, then the two can be performed in this order in pq ways.*

The truth of this theorem follows when we consider that any one of the p ways of performing the first act can be followed by any one of the q ways of performing the second act.

The following is a generalization of this theorem.

If one act can be performed in p ways, then a second in q ways, then a third in r ways, and so on, they can all be performed in this order in $pqr \cdots$ ways.

Ex. Prove this generalization for the case in which there are three acts to be performed.

104. The number of permutations of n things taken r at a time. — We are now ready to determine the number of permutations of n things taken r at a time. We shall represent this number by the symbol ${}_nP_r$, and the things to be arranged by the letters $a_1, a_2, \cdots a_n$.

For the first place in the arrangement we have the choice of any one of the n things. When this place has been filled we have the choice of any one of the remaining $n - 1$ things for the second place. When these two places have been filled, we

have the choice of any one of the remaining $n-2$ things for the third place. When k places have been filled we have the choice of any one of the remaining $n-k$ things for the $(k+1)th$ place. When all but the last, or rth, place has been filled, we have the choice of any one of the remaining $n-(r-1)$, or $n-r+1$, things for this place. Hence, by virtue of the generalization of the preliminary theorem,

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1). \qquad (1)$$

The student should observe that the first factor in this formula is the total number of things considered and that the number of factors is the number of things taken in each arrangement.

Factorial *n*. — The product of all the integers from 1 to n inclusive is called **factorial *n*** and is represented by the symbol $n!$.

If in each arrangement we take all the things; that is, if $r = n$, we get the formula

$$\begin{aligned} _nP_n &= n(n-1)\cdots(n-n+2)(n-n+1) \\ &= n(n-1)\cdots 2\cdot 1 \\ &= n! \end{aligned} \qquad (2)$$

Formula (2) is equivalent to the statement that *the number of permutations of n things taken n at a time is factorial n.*

EXERCISES

1. In how many ways can I make the trip across a lake and back if I can go over in any one of five rowboats and return in any one of four launches?

2. How many whole numbers of four unlike digits each can be formed from the digits 1, 2, 3, 4, 5, 6, 7?

3. How many signals of three flags each, in vertical order, can be made from eight flags all different?

4. How many whole numbers less than 500 and containing no repetition of digits can be formed from the digits 1, 2, 3, 4, and 5?

5. In how many ways can eight books with blue bindings and four with olive-green bindings be so arranged on a shelf that those with the same binding shall be together?

6. For a certain value of n, ${}_nP_4 = 9 \cdot {}_nP_3$. Find this value.

7. For a certain value of r, ${}_{12}P_r = 3 \cdot {}_{11}P_r$. Find this value.

8. For a certain value of r, ${}_{10}P_r = 3 \cdot {}_8P_r$. Find this value.

9. In how many ways can crops of corn, wheat, oats, and barley be raised in four fields, each field being used for a single crop?

10. In how many ways can ten keys be arranged on a ring?

SUGGESTION. — The position of one key is immaterial. The only thing that needs to be considered is the positions of the other nine relative to this one.

11. In how many different relative positions can a party of six people seat themselves at a round table?

12. Given ${}_nP_5 = 12 \cdot {}_nP_3$; solve for n.

13. A railway signal has three arms and each arm can be put into two positions besides its position of rest. How many signals can be given by it? Every position of the arms, except that in which they are all at rest, forms a signal.

14. How many kinds of single trip local passenger tickets, good either way, will a railway company need for use on a division that has ten stations?

15. In how many different relative positions can a party of five ladies and five gentlemen be seated at a round table, the ladies and gentlemen being seated alternately?

105. Distinguishable permutations of n things not all different. — The number of arrangements of the four letters of the word city is $4!$, or 24; but the number of distinguishable arrangements of the six letters of the word Canada is not $6!$.

This is due to the fact that these six letters are not all distinct. Any rearrangement of the three a's does not make a

distinguishable difference in any of the arrangements. Hence for every arrangement of these letters there would be $3!$, or 6, arrangements if the letters were all distinct. The number of distinguishable arrangements is then $\frac{6!}{3!}$, or 120.

In general, for every arrangement of n things p of which are alike, there would be $p!$ arrangements if these n things were all distinct. Hence,

Theorem. — *If of n things p are alike, the number of permutations of these things taken all together is* $\frac{n!}{p!}$.

Corollary. — *If of n things p are of one kind, q of another, r of another, and so on, the number of permutations of these things taken all together is* $\frac{n!}{p! \cdot q! \cdot r! \cdots}$.

The proof is left to the student.

EXERCISES

1. How many permutations are there of the letters of the word *permutation*?

2. How many signals can be made by displaying twelve flags one above the other, of which four are white, three blue, one red, and the rest black?

3. How many signals can be made by displaying the flags of Ex. 2 if the top flag must be red and the bottom one blue?

4. In how many ways can a row of five white balls, four red ones, and two black ones be arranged?

5. Four dimes, three quarters, and three half-dollars are to be distributed among ten boys in such a way that each boy shall receive one coin. In how many ways can this be done?

106. Combinations. — If in any set of things we are interested only in the things themselves and do not consider the order in which they may be arranged, we speak of the set as a **combination.**

There is obviously only one combination of n things taken n at a time, but there is more than one combination of n things taken r at a time, if r is less than n. In certain problems it is necessary to determine this number. We shall accordingly derive a general formula for it, and shall represent it by the symbol ${}_nC_r$.

107. Number of combinations. — Consider first the combinations of 6 things taken 4 at a time. We can determine in the following way just how many of these there are.

The 4 things of any one of these combinations can be arranged in 4! different orders and these different arrangements give rise to 4! distinct permutations of the 6 things taken 4 at a time. Moreover every permutation of the 6 things 4 at a time is related to some one of these combinations in this way. Hence,

$$ {}_6C_4 \cdot 4! = {}_6P_4, $$

and therefore

$$ {}_6C_4 = \frac{6 \cdot 5 \cdot 4 \cdot 3}{4!}. $$

In general, since there are $r!$ permutations of r things taken r at a time, it follows that for every combination of n things taken r at a time there are $r!$ distinct permutations of these n things taken r at a time. Moreover every permutation of these n things r at a time is related to some one of these combinations in this way. Hence,

$$ {}_nC_r \cdot r! = {}_nP_r, $$

and therefore

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \tag{3} $$

If we multiply both numerator and denominator of this expression for ${}_nC_r$ by $(n-r)!$ we get

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{r! \cdot (n-r)!} = \frac{n!}{r! \cdot (n-r)!}. \tag{4} $$

108. Whenever we make a combination of r things from n things, there is left a combination of $(n-r)$ things, and two

different combinations of r things from the same n things leave different combinations of $(n - r)$ things. Hence,

The number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $n - r$ at a time. That is, ${}_nC_r = {}_nC_{n-r}$.

Prove this last statement directly from Formula (4).

PROBLEMS

1. How many straight lines can be drawn through pairs of points selected from nine points no three of which are in a straight line?

2. If the lines of Ex. 1 are produced indefinitely and no two of them are parallel, how many triangles will be formed?

3. If no four of a set of fourteen points lie in one plane, how many tetrahedra are there whose vertices are in this set?

4. How many planes are determined by the points of Ex. 3?

A plane is determined by three points not in a straight line.

5. At a meeting of ten men each shakes hands with all the others. How many handshakes are there?

6. How many committees each consisting of five Republicans and four Democrats can be chosen from twenty Republicans and sixteen Democrats?

7. How many parallelograms are formed when a set of nine parallel lines is met by another set of eight parallel lines?

8. In how many ways can seven ladies and four gentlemen arrange a game of tennis, each side to consist of one lady and one gentleman?

9. In how many ways can a baseball nine be formed from sixteen players, of whom nine can play only in the infield and seven only in the outfield? Each of the infield players can play any one of the six infield positions, and each of the outfield players can play any one of the three outfield positions.

10. In how many ways can a committee of five men be selected from a group of twelve men?

11. How many sub-committees, each containing five members, of whom at least three shall be Democrats, can be formed from a committee of eight Democrats and five Republicans?

12. Given ${}_nP_r = 20$, ${}_nC_r = 10$; find n and r.

13. How many groups of three letters each can be formed from the letters of the alphabet?

14. In how many ways can a picket of five men be formed from a company of 100 soldiers?

15. How many combinations each consisting of four red balls, two white ones, and five black ones can be formed from seven red balls, six white ones, and nine black ones?

16. In how many ways can a committee of four lawyers, three merchants, and one physician be formed from the stockholders of a corporation consisting of ten merchants, eight lawyers, and five physicians?

17. How many arrangements of four consonants and three vowels can be made from eight consonants and five vowels?

109. The binomial theorem for positive integral powers. — Consider the product

$$(a+b)(a+b)(a+b)$$

of 3 equal binomial factors. The product of the first terms of all the factors, namely a^3, is a term of the product. So also is the product of the first terms of any two of the factors and the second term of the remaining factor. This is a^2b. Moreover there are three such terms in the product since the factor whose second term is used may be any one of the three factors. Hence when like terms are combined the product will contain the term $3a^2b$. It is evident from the way this term was obtained that in it the coefficient of a^2b is equal to the number of ways in which the term b can be selected from the three given factors, and this number is ${}_3C_1$. The product of the first term

of any factor and the second terms of the other factors, namely ab^2, is also in the product; and there are as many such terms as there are ways of selecting the term b from two of the three factors. Hence, when like terms are combined, the coefficient of ab^2 is ${}_3C_2$. Finally the product contains the product of the second terms of all the factors. Hence,

$$(a+b)^3 = a^3 + {}_3C_1a^2b + {}_3C_2ab^2 + b^3.$$

Consider now the product

$$(a+b)(a+b)\cdots(a+b)$$

of n equal binomial factors. The product of the first terms of all the factors, namely, a^n, is a term of the product. So also is the product of the first terms of any $n-1$ of the factors and the second term of the remaining factor. This is $a^{n-1}b$. Moreover there are n such terms in the product since the factor whose second term is used may be any one of the n factors. Hence when like terms are combined the product will contain the term $na^{n-1}b$. It is evident from the way this term was obtained that in it the coefficient of $a^{n-1}b$ is equal to the number of ways in which the term b can be selected from the n given factors, and this number is ${}_nC_1$. The product of the first terms of any $n-2$ of the factors and the second terms of the remaining factors, namely, $a^{n-2}b^2$, is also in the product; and there are as many such terms as there are ways of selecting the term b from two of the n factors. Hence when like terms are combined the coefficient of $a^{n-2}b^2$ is ${}_nC_2$. In a similar way it can be seen that in the product the coefficient of $a^{n-r}b^r$, where r is any integer from 1 to n inclusive, is ${}_nC_r$. Hence for any positive integral value of n,

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots + {}_nC_nb^n.$$

The statement of this relation is called the **binomial theorem** for positive integral powers. The right member of this identity is called the expansion of $(a+b)^n$.

If we take into account the values of ${}_nC_r$ for different values of r, as given in § 107, we can write the binomial theorem as follows:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$

$$+ \frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}b^r + \cdots + b^n. \qquad \text{(I)}$$

110. The term whose coefficient is ${}_nC_r$ is the $(r+1)$th term from the left and the $(n+1-r)$th term from the right. It is called the *general term* since any term except the first one can be obtained from it by substituting in it the proper value of r.

The term whose coefficient is ${}_nC_s$ will be as far from the right end of the expansion as ${}_nC_r$ is from the left end if

$$r+1 = n+1-s;$$

that is, if

$$s = n-r.$$

Hence the terms whose coefficients are ${}_nC_r$ and ${}_nC_{n-r}$ respectively are equidistant from the ends. But ${}_nC_r = {}_nC_{n-r}$ (§ 108). This proves that *the coefficients of any two terms equidistant from the ends of the expansion of* $(a+b)^n$ *are equal.*

111. Greatest coefficient. — Since

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

and

$${}_nC_{r-1} = \frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!},$$

it follows that

$${}_nC_r = \frac{n-r+1}{r}\,{}_nC_{r-1}.$$

Hence ${}_nC_r$ is greater than the preceding coefficient when $\frac{n-r+1}{r} > 1$; that is, when $r < \frac{n+1}{2}$. (See § 182.)

If n is odd so that we can have $r = \frac{n+1}{2}$, the two coefficients

${}_nC_{\frac{n+1}{2}}$ and ${}_nC_{\frac{n-1}{2}}$ are equal and greater than any of the other coefficients.

If n is even, so that we cannot have $r = \frac{n+1}{2}$, the greatest coefficient is ${}_nC_r$, where r is the greatest integer less than $\frac{n+1}{2}$. This integer is $\frac{n}{2}$.

Hence if n is even, ${}_nC_{\frac{n}{2}}$ is the greatest coefficient in the expansion of $(a+b)^n$.

112. The expansion of $(a+b)^n$ as given by (I) can be described in words as follows:

It contains $(n+1)$ terms.

The first term is a^n (or a^nb^0) and the last is b^n (or a^0b^n).

The second term is $na^{n-1}b$.

The exponent of a in any term after the first is one less than it is in the preceding term.

The exponent of b in any term after the first is one more than it is in the preceding term.

The coefficient of any term after the first is obtained from the preceding term by dividing the product of the coefficient of this term and the exponent of a by one more than the exponent of b.

It should be observed that the sum of the exponents of a and b in any term is n.

The expansion of $(a-b)^n$ can be obtained from that of $(a+b)^n$ by putting $-b$ in place of b, since $a-b=a+(-b)$.

Example 1.—Expand $\left(\frac{2r}{3}+\frac{s}{5}\right)^5$.

Here $a=\frac{2r}{3}$, $b=\frac{s}{5}$, and $n=5$.

$$\text{Hence } \left(\frac{2r}{3}+\frac{s}{5}\right)^5 = \left(\frac{2r}{3}\right)^5 + 5\left(\frac{2r}{3}\right)^4\frac{s}{5} + \frac{5\cdot 4}{2!}\left(\frac{2r}{3}\right)^3\left(\frac{s}{5}\right)^2$$

$$+\frac{5\cdot 4\cdot 3}{3!}\left(\frac{2r}{3}\right)^2\left(\frac{s}{5}\right)^3 + \frac{5\cdot 4\cdot 3\cdot 2}{4!}\left(\frac{2r}{3}\right)\left(\frac{s}{5}\right)^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{5!}\left(\frac{s}{5}\right)^5.$$

$$= \frac{32r^5}{243} + \frac{16r^4s}{81} + \frac{16r^3s^2}{135} + \frac{8r^2s^3}{225} + \frac{2rs^4}{375} + \frac{s^5}{3125}.$$

Example 2. — Expand $(2x-3y)^6$.

Here $a=2x$, $b=-3y$, and $n=6$.

Hence $(2x-3y)^6$

$$=(2x)^6+6(2x)^5(-3y)+\frac{6\cdot5}{2!}(2x)^4(-3y)^2$$

$$+\frac{6\cdot5\cdot4}{3!}(2x)^3(-3y)^3+\frac{6\cdot5\cdot4\cdot3}{4!}(2x)^2(-3y)^4$$

$$+\frac{6\cdot5\cdot4\cdot3\cdot2}{5!}(2x)(-3y)^5+\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6!}(-3y)^6$$

$$=64x^6-576x^5y+2160x^4y^2-4320x^3y^3+4860x^2y^4-2916xy^5+729y^6.$$

EXERCISES

Expand the following expressions by the binomial theorem.

1. $(x+y)^8$.
2. $(a-2)^4$.
3. $(3-x)^5$.
4. $(b+1)^7$.
5. $(2a+4c)^5$.
6. $(2a^2-b^2)^8$.
7. $(3a^2b+2c^3)^5$.
8. $(\sqrt{x}+\sqrt{y})^3$.
9. $\left(\frac{a^3}{b}+\frac{3b^2}{a}\right)^4$.
10. $(x+1)^{10}$.
11. $(1+x)^{10}$.
12. $(x^{\frac{1}{3}}+y^{\frac{1}{3}})^3$.
13. $(a+b)^4-(a-b)^4$.
14. $(5ab^{-3}-2x^3y^2)^5$.
15. $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)^6$.
16. $(\sqrt{a}+\sqrt[3]{b})^8$.
17. $(3a^2b^4-\frac{1}{2})^4$.
18. $(a+b+c)^3$.
19. $(m^{-3}-n^{-3})^5$.
20. $\left(x^3+\frac{1}{x^3}\right)^4$.

In each of the following exercises write the term asked for without finding any other term.

21. The 6th term of $(a+b)^9$.
22. The 6th term of $(a-b)^9$.
23. The 5th term of $(3a-2b)^8$.
24. The 4th term of $(4x^2y-5xy^2)^{10}$.

25. The 3d term of $(1-x)^7$.

26. The 8th term of $\left(2x+\frac{1}{2x}\right)^{11}$.

27. Write the third term from each end of the expansion of $(4a-7b)^9$.

28. Write the middle term of $(7a^2-4b^2)^8$.

29. What is the coefficient of x^5 in the expansion of $(1-x)^{12}$?

30. Write the first five terms of the expansion of $\left(1+\frac{1}{n}\right)^n$.

31. Are the coefficients in the expansion given in the result of the first illustrative example consistent with § 109 (I)?

Compute the values of the following correct to two decimal places:

32. $(1.1)^8$.

HINT.—$(1.1)^8 = (1+.1)^8$

$$= 1 + 8(.1) + 28(.1)^2 + 56(.1)^3 + 70(.1)^4 + 56(.1)^5 + 28(.1)^6 + 8(.1)^7 + (.1)^8$$

$$= 1 + .8 + .28 + .056 + .007 = 3.14.$$

The last four terms have been neglected since they can have no influence on the first two decimal places.

33. $(3.1)^9$. 34. $(.99)^6$. 35. $(1.01)^7$. 36. $(4.9)^6$.

CHAPTER XI

MATHEMATICAL INDUCTION

113. We consider in this chapter a method of proof, common in mathematics, of a certain class of theorems in which an integer is in some way involved.

We shall first apply the method in question to a simple theorem and shall then examine its essential features.

114. Theorem. — *The sum of the first n even integers beginning with 2 is equal to the product of n and the next larger integer.*

I. It is easy to verify that the theorem is true for small values of n.

For example, for $n = 1$, we have $2 = 1(1 + 1)$.

II. Suppose that the theorem is true for $n = r$; that is, that

$$2 + 4 + 6 + \cdots + 2r = r(r + 1). \qquad (1)$$

Adding $2(r + 1)$ to each member of this supposed equation, we get

$$2 + 4 + 6 + \cdots + 2r + 2(r + 1) = r(r + 1) + 2(r + 1)$$
$$= (r + 1)(r + 2). \qquad (2)$$

Now (2) is true if (1) is true. But (2) is merely another form of the statement that the theorem is true for $n = r + 1$. If then the theorem is true for $n = r$, it is true for $n = r + 1$. But we know from I that it is true for $n = 1$, and, therefore, it is true for $n = 1 + 1 = 2$. And since it is true for $n = 2$, it is true for $n = 2 + 1 = 3$. We can proceed in this way by successive steps until we get the proof that the theorem is true for any given integral value of n.

115. This method of proof is called **mathematical induction.** It is applicable to certain theorems in which a positive integer is in some way involved, and consists of two parts; namely, the direct verification of the theorem for the smallest admissible value of the integer and the proof that if the theorem is true for one value of the integer, it is true for the next greater value.

116. Necessity of both parts of the proof. — The formula

$$1^2 + 2^2 + \cdots + n^2 = \frac{5n^2 - 7n + 4}{2}$$

is true for $n = 1$. It is also true for $n = 2$ and $n = 3$, but it is not true for all values of n, as may be seen by putting $n = 4$.

This gives

$$1^2 + 2^2 + 3^2 + 4^2 = \frac{80 - 28 + 4}{2} = 28.$$

But this is obviously not true.

On the other hand, the formula

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1) + 5$$

is true for $n = r + 1$, provided it is true for $n = r$.

For, if the formula is true for $n = r$, we have

$$1^2 + 2^2 + \cdots + r^2 = \tfrac{1}{6}r(r+1)(2r+1) + 5,$$

and if we add $(r+1)^2$ to each member of the equation, we get

$$\begin{aligned} 1^2 + 2^2 + \cdots + r^2 + (r+1)^2 &= \tfrac{1}{6}r(r+1)(2r+1) + 5 + (r+1)^2 \\ &= \tfrac{1}{6}(r+1)(2r^2 + r + 6r + 6) + 5 \\ &= \tfrac{1}{6}(r+1)(2r^2 + 7r + 6) + 5 \\ &= \tfrac{1}{6}(r+1)(r+2)(2r+3) + 5 \\ &= \tfrac{1}{6}(r+1)(\overline{r+1}+1)(\overline{2r+1}+1) + 5. \end{aligned}$$

But this is what the original formula becomes when we put $n = r + 1$. Hence if the formula were true for $n = r$, it would be true for $n = r + 1$.

That this argument, however, does not establish the validity of the formula may readily be seen by testing the formula for $n = 1$. This gives

$$1^2 = \tfrac{1}{6} \cdot 1(1+1)(2+1)+5 = 1+5 = 6,$$

which is obviously untrue.

These two examples show that both parts of the proof as outlined are necessary in order to establish the truth of a theorem.

EXERCISES

Establish the following formulæ by mathematical induction:

1. $1+3+5+\cdots+(2n-1)=n^2$.

2. $1^2+2^2+3^2+\cdots+n^2=\tfrac{1}{6}n(n+1)(2n+1)$.

3. $1^3+2^3+3^3+\cdots+n^3=\tfrac{1}{4}n^2(n+1)^2$.

4. $a+ar+ar^2+\cdots+ar^{n-1}=\dfrac{a(1-r^n)}{1-r}$.

5. $2+2^2+2^3+\cdots+2^n=2(2^n-1)$.

6. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$.

7. $2\cdot 4+4\cdot 6+6\cdot 8+\cdots+2n(2n+2)=\dfrac{n}{3}(2n+2)(2n+4)$.

8. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots+n(n+1)(n+2)$
$$=\frac{n}{4}(n+1)(n+2)(n+3).$$

9. Prove that if n is a positive integer, $a^n - b^n$ is divisible by $a-b$.

HINT. $a^{r+1}-b^{r+1}=a(a^r-b^r)+b^r(a-b)$. The right member of this identity is divisible by $a-b$, if a^r-b^r is divisible by $a-b$.

10. Prove that if n is a positive integer, $a^{2n}-b^{2n}$ is divisible by $a+b$.

117. The binomial theorem for positive integral powers. — The student is familiar with the fact that

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and that $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Now these relations are special cases of a general theorem which tells us how to write out, or expand, any positive integral power of a binomial. This theorem is known as the **binomial theorem.**

It says that if n is any positive integer

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$

$$+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}$$

$$+ \frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}b^r + \cdots + b^n.$$

The right member of this formula can be described as follows:

It contains $(n+1)$ *terms.*

The first term is a^n *(or* a^nb^0*) and the last is* b^n *(or* a^0b^n*).*

The second term is $na^{n-1}b$.

The exponent of a *in any term after the first is one less than it is in the preceding term.*

The exponent of b *in any term after the first is one more than it is in the preceding term.*

The coefficient of any term after the first is obtained from the preceding term by dividing the product of the coefficient of this term and the exponent of a *by one more than the exponent of* b.

It should be observed that the sum of the exponents of a and b in any term is n.

118. Proof of the binomial theorem. — We shall prove the theorem by mathematical induction.

In order to do this we observe in the first place that the theorem is obviously true for $n = 1$. (The student can also verify directly that it is true also for $n = 2$ and $n = 3$, although this is not necessary for the argument.) In the second place we assume that it is true for $n = m^*$; that is, we assume the truth of the statement that

$$(a+b)^m = a^m + ma^{m-1}b + \cdots + \frac{m(m-1)\cdots(m-r+1)}{r!}a^{m-r}b^r + \cdots + mab^{m-1} + b^m.$$

Then we multiply each member of this identity by $a + b$. This gives us

$$(a+b)^{m+1} =$$

$$a^{m+1} + ma^m b + \cdots + \frac{m(m-1)\cdots(m-r+1)}{r!}a^{m-r+1}b^r + \cdots + ab^m$$

$$+ a^m b + \cdots + \frac{m(m-1)\cdots(m-r+2)}{(r-1)!}a^{m-r+1}b^r + \cdots + mab^m + b^{m+1}$$

$$= a^{m+1} + (m+1)a^m b + \cdots + \frac{(m+1)m(m-1)\cdots(m-r+2)}{r!}a^{m-r+1}b^r$$

$$+ \cdots + (m+1)ab^m + b^{m+1}.$$

Now this expansion for $(a + b)^{m+1}$ is true provided that the expansion for $(a + b)^m$ with which we started is true. Moreover this expansion for $(a + b)^{m+1}$ is in accordance with the binomial theorem. Hence, if the theorem is true for $n = m$, it is true for $n = m + 1$. But we have already observed that it is true for $n = 1$. It is therefore true for all positive integral values of n.

In deriving the expansion of $(a + b)^{m+1}$ we made use of the fact that

* We do not say $n = r$ because we are here using r for another purpose.

$$\frac{m(m-1)\cdots(m-r+1)}{r!}+\frac{m(m-1)\cdots(m-r+2)}{(r-1)!}$$

$$=\frac{m(m-1)\cdots(m-r+1)+rm(m-1)\cdots(m-r+2)}{r!}$$

$$=\frac{m(m-1)\cdots(m-r+2)[(m-r+1)+r]}{r!}$$

$$=\frac{(m+1)m(m-1)\cdots(m-r+2)}{r!}.$$

119. The general term. — The term

$$\frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}b^r$$

is the $(r+1)th$ term in the expansion of $(a+b)^n$. It is called the **general term** of this expansion since it can be made equal to any term, except the first one, by assigning to r a suitable value.

The expansion of $(a-b)^n$ can be obtained from that of $(a+b)^n$ by putting $-b$ in place of b, since $a-b=a+(-b)$.

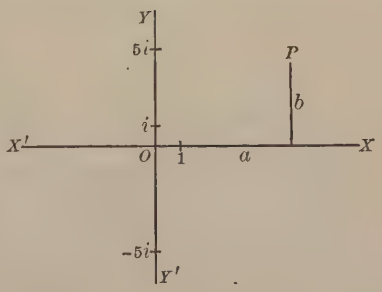
For exercises on the binomial theorem, see those in § 112.

CHAPTER XII

COMPLEX NUMBERS

120. In considering negative numbers for the first time it was found helpful to associate with every point of a given line a number which represented its distance from a given point (or origin) on the line. In order to distinguish between two points equally distant from the origin but on opposite sides of it we agreed to represent the distances of points on one side of the origin (say the right side) by positive numbers and the distances of points on the opposite side of the origin by negative numbers. The numbers that represent these points were called **real** numbers.

Now a similar but more extensive association of numbers with points will serve to make clear in a simple way the essential nature of a more complicated kind of numbers that are of great importance in mathematics; namely, the so-called **imaginary**, or **complex**, numbers.



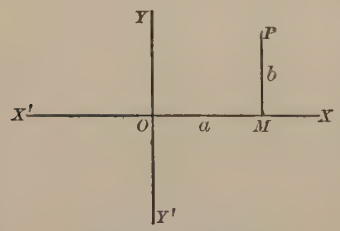
Consider two perpendicular lines $X'X$ and $Y'Y$ that intersect at 0, and let the real numbers represent the points of $X'X$, the origin being taken at 0.

We want to represent the points of the plane that are not on $X'X$ by symbols which we can use in a way that resembles as closely as possible the use of the real numbers (the symbols that represent the points of $X'X$) in the operations of addition, subtraction, multiplication, and division. We shall call these symbols *numbers*.

121. In Chapter V we represented the points of $Y'Y$, as well as those of $X'X$, by real numbers, but for our present purpose we must represent the points of the plane that are not on $X'X$ by numbers different from the real numbers, since every real number represents a point on $X'X$, and we do not want any number to represent two points.

The number by which we shall represent the point on $Y'Y$ the unit's distance above 0 will be called i. Any point on $Y'Y$ whose distance from 0 is a will be represented by the number ia. If a is positive, the point represented by ia is above 0, and if a is negative, the point is below 0. Thus, the point represented by $5i$ (or, $i \cdot 5$) is on $Y'Y$ five units above 0, and the point represented by $-5i$ is on $Y'Y$ five units below 0.

If we should mark on $X'X$ the point M represented by the number a and should then proceed to lay off the point ib in the way just described, with the exception that we use the point M instead of 0 for origin, we should get the point P which is b units from M and above or below it according as b is positive or negative.



We shall say that the number representing P is the sum of a and ib and shall indicate it by the symbol $a + ib$. This gives us a definition of the sum of a real number and a number of the form ib that is closely analogous to the definition of the sum of two real numbers given in § 4.

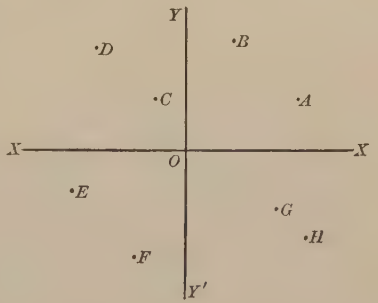
122. Let now P be any point of the plane. Draw the perpendicular PM from P to $X'X$. If $OM = a$ and $MP = b$ the number that represents P is $a + ib$. We have at hand therefore a system of numbers that will represent all the points of the plane, and conversely, every number of this form; namely, $a + ib$, represents a point of the plane.

123. Equal numbers. — We shall agree to say that two numbers are equal when they represent the same point. Hence if $a + ib = c + id$, we must have $a = c$, and $b = d$. Conversely, if these last two inequalities hold, the numbers $a + ib$ and $c + id$ are equal.

If the point represented by $a + ib$ is on $X'X$, $b = 0$. If the point is the origin, $a = 0$, $b = 0$; that is, if $a + ib = 0$, $a = 0$, and $b = 0$.

EXERCISES

1. Write the numbers that represent the points designated in the accompanying figure.

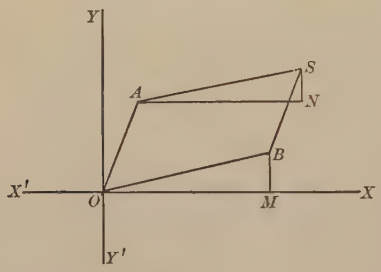


2. Plot the points represented by the following numbers:

$3 + i$, $3 - i$, $-3 + i$, $5i$,
$-i$, $1 + i$, $-1 - 3i$,
$4 - 2i$, 6, $-3 + 4i$,
$-5 - 2i$.

124. It was stated in § 120 that we wanted to be able to use these new numbers in a way resembling as closely as possible the use of real numbers in the operations of addition, subtraction, multiplication, and division. Now the essential point in the use of real numbers is that it is governed by the assumptions of Chapter I. In the following paragraphs we shall accordingly describe combinations of these new numbers which we shall call addition, subtraction, multiplication, and division respectively, and which will also be governed by these assumptions. It will be seen that these new operations are perfectly natural extensions of the corresponding operations with real numbers. We have already described the addition of a real number and a number of the form ib.

125. Addition and subtraction. — By the sum of the numbers $x_1 + iy_1$ and $x_2 + iy_2$ representing the points A and B respectively, we mean the number $x + iy$ representing the point S which is obtained by laying off B from A, instead of from 0, as origin.



Thus, we draw AN equal, and parallel, to OM, and then NS equal and parallel to MB.

The student can readily verify that S is the fourth vertex of the parallelogram two of whose sides are OA and OB, except when O, A, and B lie on a straight line. He can also see that

$$x = x_1 + x_2 \text{ and } y = y_1 + y_2.$$

Hence, $$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

By the difference of $x_1 + iy_1$ and $x_2 + iy_2$, or $(x_1 + iy_1) - (x_2 + iy_2)$, we mean the number $x + iy$ such that

$$\begin{aligned} x_1 + iy_1 &= (x + iy) + (x_2 + iy_2) \\ &= (x + x_2) + i(y + y_2). \end{aligned}$$

It follows from § 123 that

$$x_1 = x + x_2 \text{ and } y_1 = y + y_2.$$

Hence, $$x = x_1 - x_2 \text{ and } y = y_1 - y_2,$$

and therefore $$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2).$$

This difference can be obtained graphically by forming the sum of $x_1 + iy_1$ and $-x_2 + i(-y_2)$.

EXERCISES

In each of the following exercises represent graphically the numbers in the parentheses, and their sum or difference, as the case may be:

1. $(1 + 2i) + (5 - 5i)$.
2. $2i - 4i$.
3. $(1 - 2i) - (2 + 5i)$.
4. $3 - (6 - 2i)$.
5. $3i - 2$.
6. $(6 + 2i) + (5 - 4i)$.

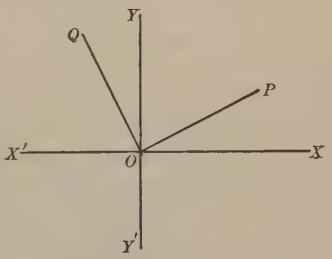
7. $(7-i)-(7+3i)$.
8. $(5-4i)+(5-4i)$.
9. $(-3-3i)+4i$.
10. $3-(6-6i)$.
11. $(8+4i)-(8+4i)$.
12. $(-3\frac{1}{2}+2i)-(-3\frac{1}{2}+i)$.
13. $(\sqrt{2}+\sqrt{3}i)+(\sqrt{2}-\sqrt{3}i)$.
14. $(4i+3)+(4+3i)$.
15. $(2+6i)-(2+6i)$.
16. $(3-2i)+(2i-3)$.
17. $(7-2i)+(7+2i)$.
18. $(1+\sqrt{5}i)+(2+2\sqrt{5}i)$.

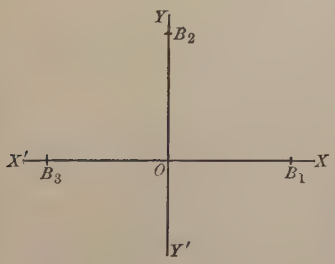
19. What is the relation between the points represented by $3+4i$ and $-3-4i$ respectively? Plot these points and also the points $3-4i$ and $-3+4i$.

20. Under what circumstances is the sum of two complex numbers a real number?

21. Apply the definitions of addition and subtraction in this article to two real numbers and compare the results with those obtained by applying the definitions of §§ 4 and 6.

126. Multiplication. — By the product of i and the number representing any point P we mean the number representing the point to which P is brought by revolving OP counterclockwise around O through an angle of 90°.





Thus, if the angle POQ is a right angle and $OP = OQ$, the number representing Q is equal to the number representing P multiplied by i.

In the figure let $OB_1 = OB_2 = OB_3$. If b represents the point B_1 on $X'X$, the product of i and b represents B_2 on $Y'Y$. But we have already represented this point by ib. Hence we say that ib is the product of i and b.

The product of i and ib represents the point B_3 on $X'X$. But $-b$ also represents this point. Hence,

$$i \cdot ib = -b,$$

or

$$i^2 b = -b.$$

This is equivalent to saying that

$$i^2 = -1.$$

127. When we consider that the product of two positive, or two negative, numbers is positive, it seems strange that we can have a number like i whose square is a negative number; or, in other words, that we can have a square root of a negative number. The explanation is that this number is neither positive nor negative, since it represents a point not on the axis $X'X$. It is a new kind of number and it ought not to be surprising that it should have properties not possessed by the old numbers.

If we had never heard of negative numbers, it would seem just as strange to talk about a number that is equal to 3 minus 5 as it does to talk about a number whose square is -1.

128. Definitions.— Numbers of the form ib, where b is a real number, different from zero, are called **pure imaginaries.** Numbers of the form $a + ib$, where a and b are real numbers and b is not equal to zero, are called **imaginary numbers.**

This term, *imaginary*, was applied at a time when it was supposed that these numbers had no existence and were not real. But with the conception of numbers that has been set forth here we see that they are just as real as the points of a plane, which they represent, and therefore just as real as the so-called real numbers.

Since imaginary numbers are formed by a combination of two essentially different kinds of units; namely, 1 and i, they are more properly called **complex numbers.**

In the complex number $a + ib$, a is called the *real* part and ib the *imaginary* part.

In a real number the imaginary part is zero and in a pure imaginary the real part is zero.

Complex numbers that differ only in the sign of their imaginary parts are said to be **conjugate to each other.**

Thus $4 + 3i$ and $4 - 3i$ are conjugate, as are also $a + ib$ and $a - ib$.

A real number is its own conjugate.

129. Product of two complex numbers. — We are now in a position to define what we mean by the product of two complex numbers like $a + ib$ and $c + id$.

We agree to say that

$$(a + ib)(c + id) = ac + ibc + iad + i^2bd = (ac - bd) + i(bc + ad).$$

It will be observed that this product is just what it would be if the numbers a, b, c, d, and i were all real, with the exception that it makes use of the fact that $i^2 = -1$.

130. Theorem. — *The sum and the product of two conjugate complex numbers are real numbers.*

For $$(a + ib) + (a - ib) = 2a,$$

and $$(a + ib)(a - ib) = a^2 + b^2.$$

131. Division of complex numbers. — We define the division of $a + ib$ by $c + id$ to be the operation of finding the number $x + iy$ such that $a + ib = (x + iy)(c + id)$.

The indicated quotient $\dfrac{a + ib}{c + id}$ can be changed to the form $x + iy$ by multiplying both the numerator and the denominator of the fraction by the conjugate of the denominator. Thus

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + i\,\frac{bc - ad}{c^2 + d^2}.$$

This process justifies the statement of Assumption XII of Chapter I as applied to complex numbers.

132. We have now defined the four fundamental operations of addition, subtraction, multiplication, and division, as applied to complex numbers, in a way which can be shown to be consistent with the assumptions of Chapter I. Moreover when applied to numbers whose imaginary parts are zero, these operations are the same as the operations of addition, subtraction, multiplication, and division respectively, as applied to real numbers.

133. Principal square root of negative numbers. — Since i^2 and $(-i)^2$ are both equal to -1, i and $-i$ are square roots of -1. We shall call i the **principal square root of** -1 and represent it by $\sqrt{-1}$. Thus $i = \sqrt{-1}$.

By the *principal square root* of any negative number such as $-a$, where a is a positive number, we mean the product of i and the principal square root of a (see § 55). We shall use the symbol $\sqrt{-a}$ to indicate this principal square root.

Thus $$\sqrt{-a} = \sqrt{a} \cdot i = \sqrt{a}\,\sqrt{-1}.$$

If b is also a positive number, then

$$\sqrt{-a} \cdot \sqrt{-b} = \sqrt{a} \cdot i \cdot \sqrt{b} \cdot i = \sqrt{ab} \cdot i^2 = -\sqrt{ab}.$$

The student is warned against saying

$$\sqrt{-a} \cdot \sqrt{-b} = \sqrt{-a \cdot -b} = \sqrt{ab}.$$

EXERCISES

Reduce each of the following expressions to the standard form $a + ib$.

1. $(3 - 2i)(4 - 5i)$.
2. $(2 + i)^2$.
3. i^3.
4. i^4.
5. i^5.
6. i^{14}.
7. $\dfrac{1}{i}$.
8. $(2 + \sqrt{-3})(7 - \sqrt{-4})$.

HINT.

$2 + \sqrt{-3} = 2 + \sqrt{3}\,i$,

$7 - \sqrt{-4} = 7 - 2i$.

9. $\left(\frac{-1+\sqrt{-3}}{2}\right)^2$.

10. $\left(\frac{-1+\sqrt{-3}}{2}\right)^3$.

11. $\frac{-1+\sqrt{-3}}{2} \cdot \frac{-1-\sqrt{-3}}{2}$.

12. $\frac{2}{-1+\sqrt{3}\,i}$.

13. $(2-i)(2+i)(3+2\,i)$.

14. $\frac{1+i}{1-i}$.

15. $(\sqrt{2}+\sqrt{5}\,i)(i\sqrt{2}+\sqrt{5})$.

16. $(\sqrt{x}+\sqrt{y}\,i)(i\sqrt{x}+\sqrt{y})$.

17. $\left(\frac{1-i}{1+i}\right)^3$.

18. $5\,i \cdot -8\,i$.

19. $\frac{5-7\,i}{3+i}+\frac{6+i}{2+i}$.

20. $\frac{5-7\,i}{3+i} \cdot \frac{6+i}{2+i}$.

21. $\frac{1-\sqrt{3}\,i}{2} \cdot \frac{-1+\sqrt{3}\,i}{2}$.

22. $\frac{\sqrt{-7}+\sqrt{-8}}{\sqrt{-9}+\sqrt{-2}}$.

23. $\frac{3}{\sqrt{3}+\sqrt{2}\,i}$.

24. $\left(\frac{1-\sqrt{-3}}{2}\right)^6$.

25. What is the relation between the points represented by two conjugate numbers?

Solve the following equations, plot the points represented by the roots, and show graphically that the sum of the roots of each equation is zero.

26. $x^3-1=0$.

27. $x^3+1=0$.

28. $x^4-1=0$.

29. $x^6-1=0$.

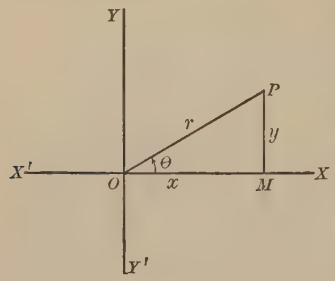
30. Show that $1+i$ is a root of the equation

$$2\,x^3-x^2-2\,x+6=0.$$

31. Show that $\frac{1}{2}-\frac{2}{3}\,i$ is a root of the equation

$$36\,x^3+36\,x^2-47\,x+50=0.$$

134. The polar representation of complex numbers. — There is another method of representation of complex numbers that is sometimes useful. Consider the complex number $x + iy$ that represents the point P of the figure.



If $OP = r$ and $\angle XOP = \theta$,

then
$$x = r\cos\theta, \quad (1)$$
$$y = r\sin\theta, \quad (2)$$
$$x^2 + y^2 = r^2, \quad (3)$$

and therefore

$$x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

This representation of a complex number in terms of r and θ is called the **polar representation** of the number. The angle θ is called the **amplitude**, or **argument**, and the distance OP, or r, is called the **modulus**, or **absolute value**, of the complex number.

It follows from formula (3) that

$$r = \sqrt{x^2 + y^2}$$

is the modulus of the number $x + iy$. It is always considered positive.

It follows from (1) and (2) that the argument θ of $x + iy$ is given by the formulæ

$$\theta = \cos^{-1}\frac{x}{r} = \cos^{-1}\frac{x}{\sqrt{x^2 + y^2}},$$

$$\theta = \sin^{-1}\frac{y}{r} = \sin^{-1}\frac{y}{\sqrt{x^2 + y^2}}.$$

For example, the modulus of $2 - 3i$ is $r = \sqrt{13}$ and the argument is

$$\theta = \cos^{-1}\frac{2}{\sqrt{13}} = \sin^{-1}\frac{-3}{\sqrt{13}}.$$

135. Theorem. — *The modulus of the product of two numbers is equal to the product of their moduli and the argument of the product is equal to the sum of their arguments.*

For

$$\begin{aligned} &r(\cos\theta + i\sin\theta)\cdot r'(\cos\theta' + i\sin\theta') \\ &= (r\cos\theta + ir\sin\theta)(r'\cos\theta' + ir'\sin\theta') \\ &= rr'\cos\theta\cos\theta' - rr'\sin\theta\sin\theta' \\ &\quad + i(rr'\sin\theta\cos\theta' + rr'\cos\theta\sin\theta') \\ &= rr'[(\cos\theta\cos\theta' - \sin\theta\sin\theta') + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')] \\ &= rr'[\cos(\theta+\theta') + i\sin(\theta+\theta')]. \end{aligned}$$

136. Since

$$\frac{r(\cos\theta + i\sin\theta)}{r'(\cos\theta' + i\sin\theta')} = \frac{r(\cos\theta + i\sin\theta)(\cos\theta' - i\sin\theta')}{r'(\cos\theta' + i\sin\theta')(\cos\theta' - i\sin\theta')}$$

$$= \frac{r[\cos(\theta-\theta') + i\sin(\theta-\theta')]}{r'(\cos^2\theta' + \sin^2\theta')} = \frac{r}{r'}[\cos(\theta-\theta') + i\sin(\theta-\theta')],$$

we have the

Theorem. — *The modulus of the quotient of two numbers is equal to the quotient of their moduli and the argument of the quotient is equal to the argument of the dividend minus the argument of the divisor.*

137. De Moivre's Theorem. — If the two factors in § 135 are equal, we have

$$[r(\cos\theta + i\sin\theta)]^2 = r^2(\cos 2\theta + i\sin 2\theta).$$

This is a special case of the following more general formula

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta). \qquad (I)$$

The statement of this relation is known as **De Moivre's Theorem.** In the case that n is a positive integer this theorem can be proved by means of mathematical induction in connection

with the theorem of § 135. The details of the proof are left to the student. He should go through them with great care.

The proof of (I) when n is the reciprocal of a positive integer, say $\frac{1}{m}$, is as follows:

Let $\theta = m\phi.$

Then

$$(\cos\theta + i\sin\theta)^{\frac{1}{m}} = (\cos m\phi + i\sin m\phi)^{\frac{1}{m}} = [(\cos\phi + i\sin\phi)^m]^{\frac{1}{m}}$$

$$= \cos\phi + i\sin\phi = \cos\frac{1}{m}\theta + i\sin\frac{1}{m}\theta.$$

Hence, $$[r(\cos\theta + i\sin\theta)]^{\frac{1}{m}} = r^{\frac{1}{m}}(\cos\frac{1}{m}\theta + i\sin\frac{1}{m}\theta).$$

Formula (I) also holds for all rational values of n.

138. Roots of numbers. — Suppose that

$$x + iy = r(\cos\theta + i\sin\theta).$$

Then by formula (I) of the preceding article, provided that m is any integer,

$$\left[r^{\frac{1}{n}}\left(\cos\frac{\theta + m\cdot 360^\circ}{n} + i\sin\frac{\theta + m\cdot 360^\circ}{n}\right)\right]^n$$

$$= r[\cos(\theta + m\cdot 360^\circ) + i\sin(\theta + m\cdot 360^\circ)]$$

$$= r(\cos\theta + i\sin\theta) = x + iy.$$

It should be remembered that $r^{\frac{1}{n}}$ represents the principal nth root of r.

Hence any number of the form

$$r^{\frac{1}{n}}\left(\cos\frac{\theta + m\cdot 360^\circ}{n} + i\sin\frac{\theta + m\cdot 360^\circ}{n}\right), \qquad (1)$$

where m is an integer, is an nth root of

$$r(\cos\theta + i\sin\theta), \text{ or } x + iy.$$

If m is any integer (positive or negative), and we divide m by n, we shall get a quotient q and a remainder s. It may be that s will be zero, but in any case we can take q so that s will

be less than n, and not less than 0. Then

$$m = qn + s,$$

and $$\frac{\theta + m \cdot 360^\circ}{n} = \frac{\theta + (qn+s)360^\circ}{n} = \frac{\theta + s \cdot 360^\circ}{n} + q \cdot 360^\circ.$$

But we know from trigonometry that two angles which differ by a multiple of 360° have the same sines and cosines. Hence

$$\cos\left(\frac{\theta + s \cdot 360^\circ}{n} + q \cdot 360^\circ\right) = \cos\left(\frac{\theta + s \cdot 360^\circ}{n}\right)$$

and $$\sin\left(\frac{\theta + s \cdot 360^\circ}{n} + q \cdot 360^\circ\right) = \sin\left(\frac{\theta + s \cdot 360^\circ}{n}\right).$$

If m_1 and m_2 are any two values of m and if

$$\cos\frac{\theta + m_1 360^\circ}{n} = \cos\frac{\theta + m_2 360^\circ}{n} \text{ and } \sin\frac{\theta + m_1 360^\circ}{n}$$

$$= \sin\frac{\theta + m_2 360^\circ}{n}, \text{ then } \frac{\theta + m_1 360^\circ}{n} \text{ can differ from } \frac{\theta + m_2 360^\circ}{n}$$

only by a multiple of 360°, since two angles which have the same sines and cosines can differ only by a multiple of 360°. But this difference is $\frac{(m_1 - m_2)360^\circ}{n}$ and is not a multiple of 360° when m_1 and m_2 are unequal integers less than n and not less than 0.

Therefore the following values of m give all the distinct numbers of the form (1): $0, 1, 2, \cdots, n-1$. This proves the following

Theorem. — *If n is a positive integer, any number different from zero has exactly n distinct nth roots.*

Since all the nth roots have the same modulus, and since the argument of any one of them increased by $\frac{360^\circ}{n}$ is the argument of another one, we see that the points represented by the nth roots of $x + iy$ all lie on the circumference of a circle whose center is at the origin and divide this circumference into n equal parts.

EXERCISES

Find the modulus and the argument of each of the following numbers and plot the number.

1. $2+5i$.
2. $-3i$.
3. $6-\sqrt{-2}$.
4. 8.
5. $2+i$.
6. $1-i$.
7. i^3.
8. $(7+\sqrt{-10})(3+\sqrt{-4})$.
9. -16.
10. $\dfrac{3+2i}{3-2i}$.
11. $5(\cos 30^\circ + i \sin 30^\circ)$.
12. $2i\left(\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right)$.

Simplify the following products:

13. $2(\cos 10^\circ + i \sin 10^\circ) \cdot 5(\cos 25^\circ + i \sin 25^\circ)$.
14. $-5(\cos 30^\circ + i \sin 30^\circ) \cdot (\cos 15^\circ + i \sin 15^\circ)$.
15. $(1+i)^2$.
16. $7i\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)$.
17. $\left(\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}\right)$.
18. $[4(\cos 20^\circ + i \sin 20^\circ)]^5$.
19. $[\frac{1}{2}(\cos 180^\circ + i \sin 180^\circ)]^2$.
20. $[3(\cos 60^\circ + i \sin 60^\circ)]^6$.
21. $(\cos 30^\circ + i \sin 30^\circ)^3$.
22. $[2(\cos 40^\circ + i \sin 40^\circ)]^3$.
23. $[4(\cos 120^\circ + i \sin 120^\circ)]^3$.

24. Find all the cube roots of $1+i$.

Here
$$r = \sqrt{1+1} = \sqrt{2},$$
$$\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}.$$
Hence,
$$\theta = 45^\circ,$$
and
$$1 + i = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ).$$
$$\sqrt[3]{1+i} = \sqrt[6]{2}\left(\cos\frac{45^\circ + m \cdot 360^\circ}{3} + i \sin\frac{45^\circ + m \cdot 360^\circ}{3}\right).$$

We shall get all the cube roots of $1+i$ by giving to m in this expression the successive values 0, 1, 2.

For $m = 0$, we get $\sqrt[6]{2}(\cos 15^\circ + i \sin 15^\circ)$.
For $m = 1$, we get $\sqrt[6]{2}(\cos 135^\circ + i \sin 135^\circ)$.
For $m = 2$, we get $\sqrt[6]{2}(\cos 255^\circ + i \sin 255^\circ)$.

25. Find and plot all the cube roots of $-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$.

26. Find and plot all the fourth roots of $\frac{1+i}{1-i}$.

27. Find and plot all the cube roots of $-i$.

28. Solve the equation $x^5-32=0$, and plot the roots.

Here $x^5=32$.

Now $32=32(\cos 0^\circ+i\sin 0^\circ)$.

Hence $\sqrt[5]{32}=2\left(\cos\frac{0^\circ+m\cdot 360^\circ}{5}+i\sin\frac{0^\circ+m\cdot 360^\circ}{5}\right)$,

where $m=0, 1, 2, 3, 4$.

Hence the roots are 2, $2(\cos 72^\circ+i\sin 72^\circ)$,
$2(\cos 144^\circ+i\sin 144^\circ)$,
$2(\cos 216^\circ+i\sin 216^\circ)$,
$2(\cos 288^\circ+i\sin 288^\circ)$.

Solve the following equations and plot the roots:

29. $x^3-1=0$. 30. $x^3+1=0$.

Compare the method of solution used in Exs. 29 and 30 with that used in Exs. 26, 27, § 133.

31. $x^4-16=0$. 32. $x^5-1=0$. 33. $x^3+27=0$.

34. What is the relation between the roots of Equation 30 and those of Equation 33?

Simplify each of the following quotients:

35. $\frac{1}{i}$.

36. $\frac{1-i}{1+i}$.

37. $\frac{\sqrt{2}+i\sqrt{2}}{\sqrt{3}+i}$.

38. $\frac{\frac{1}{2}+\frac{i}{2}\sqrt{3}}{-i}$.

39. $\frac{5+\sqrt{-2}}{3-2\sqrt{-4}}$.

40. $\frac{6(\cos 60^\circ+i\sin 60^\circ)}{3(\cos 20^\circ+i\sin 20^\circ)}$.

41. $\frac{1}{\cos 40^\circ-i\sin 40^\circ}$.

CHAPTER XIII

THEORY OF EQUATIONS

139. The student is familiar with the solution of equations of the first and second degrees in one unknown; that is, he knows how to identify numbers described by such equations.

It happens, however, that many of the descriptions met with in the applications of mathematics are in the form of equations of higher degrees, and it is far more difficult to identify numbers from such descriptions than from linear or quadratic equations. We shall make no attempt to find all the roots of such equations, but shall confine our attention to the problem of finding their real roots.

140. This latter problem naturally divides itself into two parts:

1. The problem of finding the rational roots.
2. The problem of finding the approximate values of irrational real roots.

This is the most important problem considered in this book and the purpose of this chapter is to show how to solve it. Everything given in this chapter has a direct bearing on this solution.

141. We shall use such symbols as $f(x)$, $F(x)$, and $Q(x)$ to represent polynomials in x. These symbols are read "f of x," "large F of x," and "Q of x" respectively.

If $f(x)$ stands for $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$; that is, if

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

we shall use the symbol $f(c)$ to stand for the number obtained by putting c for x in this polynomial.

Thus if $f(x) \equiv 4x^3 + x^2 - 5x + 2,$

then $f(2) \equiv 4 \cdot 2^3 + 2^2 - 5 \cdot 2 + 2 = 28,$

and $f(0) \equiv 4 \cdot 0 + 0 - 5 \cdot 0 + 2 = 2.$

EXERCISES

Determine n and $a_0, a_1, \cdots, a_n$ in each of the following cases:

1. $f(x) \equiv -2x^4 + 7x^3 - 3x^2 - 5x - 6.$
2. $f(x) \equiv x^3 + x^2 - 1.$
3. $f(x) \equiv 5x^4 + 3x^2 + 10x.$
4. $f(x) \equiv x^5 - x^4 + 2x^3 + 5.$
5. $f(x) \equiv x^6 - 1.$
6. $f(x) \equiv (x+1)(x+2)(x+3).$
7. In each of the preceding exercises find $f(1)$, $f(-2)$, and $f(0)$.

142. Remainder Theorem. — *When $f(x)$ is divided by $x - r$ the remainder is $f(r)$.*

This remainder must be a constant since it is of lower degree in x than the divisor, which is of the first degree. The quotient must be a polynomial in x. We shall represent it by $Q(x)$ and the remainder by R.

Then $$f(x) \equiv Q(x)(x - r) + R.$$

If in this identity we put $x = r$, we get

$$f(r) = Q(r)(r - r) + R.$$

But $$Q(r)(r - r) = 0, \text{ since } r - r = 0.$$

Hence $$R = f(r).$$

The student should observe that the expression for R does not contain x and that therefore R is the same number, no matter what value is given to x.

Corollary. — *If r is a root of the equation $f(x) = 0$, then $x - r$ is a factor of $f(x)$, and conversely.*

For, by the theorem, the remainder obtained in dividing $f(x)$ by $x - r$ is $f(r)$, and by hypothesis $f(r) = 0$.

Conversely, if $f(x) \equiv Q(x)(x-r)$, then $f(r) \equiv Q(r)(r-r) = 0$, and r is a root of the equation $f(x) = 0$. (See § 65.)

This corollary is sometimes referred to as the **Factor Theorem.**

EXERCISES

Perform the following divisions and check your work by means of the *Remainder Theorem.*

1. $x^3 + 4x^2 + 3x - 10$ by $x - 2$.
2. $2x^5 - 7x^4 - x + 4$ by $x + 1$.
3. $x^4 + x^3 + x^2 + x + 1$ by x.
4. $7x^5 + 2x^4 + 8x^3 - 6x^2 - 5x + 4$ by $x - 1$.
5. $x^6 + 1$ by $x + 3$.
6. $x^4 - x^2 + 1$ by $x - 1$.
7. $x^3 - 6x^2 - 3x + 2$ by $x + .5$.
8. $3x^4 + 5x^2 - 6x - 2$ by $x + 2$.

143. Synthetic division. — In looking for the roots of the equations considered in this chapter we shall have frequent occasion to find the values of polynomials in x for given values of x. The Remainder Theorem tells us that these values are equal to the remainders obtained by dividing the polynomials by expressions of the form $x - r$. It is therefore important to be able to find these remainders as quickly as possible.

Consider the process of dividing $2x^3 - 5x^2 + 3x - 4$ by $x - 3$.

$$
\begin{array}{l|l}
2x^3 - 5x^2 + 3x - 4 & x - 3 \\
\underline{2x^3 - 6x^2} & 2x^2 + x + 6 \\
x^2 + 3x & \\
\underline{x^2 - 3x} & \\
6x - 4 & \\
\underline{6x - 18} & \\
14 &
\end{array}
$$

If we wish to shorten the work of this division as much as possible, we observe that it would be sufficient to write merely the coefficients of the various powers of x in the dividend and

the quotient and that we could omit the first term of the divisor without causing confusion, since all the divisors we are considering have the same first term. Moreover only the first term of each partial remainder need be brought down. Then our work could be arranged as follows:

$$\begin{array}{l|l} 2-5+3-4 & -3 \\ \underline{2-6} & \quad 2+1+6 \\ \quad 1 & \\ \quad \underline{1-3} & \\ \quad\quad 6 & \\ \quad\quad \underline{6-18} & \\ \quad\quad\quad 14 & \end{array}$$

We note further that it is unnecessary to write the coefficients of the quotient, since these are equal respectively to the first (or left hand) coefficients of the dividend and the successive remainders. The coefficients of the first terms of the successive partial products may also safely be omitted. The arrangement of the work will then be as follows:

$$\begin{array}{l|l} 2-5+3-4 & -3 \\ \underline{-6} & \\ \quad 1 & \\ \quad \underline{-3} & \\ \quad\quad 6 & \\ \quad\quad \underline{-18} & \\ \quad\quad\quad 14 & \end{array}$$

We could put 3 in place of -3 if we remember to add the partial products instead of subtracting them. It is desirable to do this because what we are really trying to find is the value of the polynomial when 3 is put in place of x. Moreover all the partial products can be written on one line. The work can therefore be arranged as follows:

$$\begin{array}{l|l} 2-5+3-\ 4 & 3 \\ \quad +6+3+18 & \\ \hline 2+1+6+14 & \end{array}$$

The successive terms in the last line reading from left to right, except the last one, are the coefficients of the descending powers of x in the quotient. The last term is the remainder.

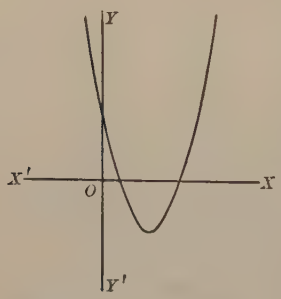
Thus if we divide $2x^3 - 5x^2 + 3x - 4$ by $x - 3$ we get the quotient $2x^2 + x + 6$ and the remainder 14. Hence this polynomial equals 14 when 3 is put for x.

This shortened form of division is known as **Synthetic Division.**

144. Rule for synthetic division. — *In order to divide the polynomial $f(x)$ by $x - r$, arrange $f(x)$ according to descending powers of x. If $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, supply zeros in the places of the coefficients of the missing terms and write the successive coefficients in a row beginning with a_0.*

Bring down a_0, multiply it by r and add the product to a_1; multiply the resulting sum by r and add the product to a_2. Continue this process of multiplying each sum by r and adding the product to the next coefficient until a product has been added to the last coefficient. The last resulting sum is the remainder.

The numbers in the row of sums, except the last one, are the coefficients of the successive terms of the quotient beginning with the term containing x^{n-1}.



145. Application of synthetic division. — This method of finding the value of $f(x)$ for a given value of x can be used to good advantage in finding points on the loci of equations of the form $y = f(x)$. If $f(a) = b$, the point (a, b) is on the locus of the equation $y = f(x)$. Thus the locus of $y = x^2 - 6x + 5$ passes through the points $(0, 5)$, $(1, 0)$, $(2, -3)$, etc.

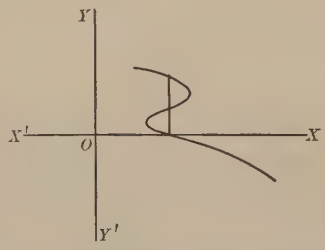
146. Graphical solution of equations. — The abscissæ of the points that are common to the locus of the equation $y = f(x)$ and the x-axis are the real roots of the equation $f(x) = 0$. This

suggests at once that we can find the real roots of $f(x)=0$ graphically by drawing the locus of the equation $y=f(x)$ and measuring the abscissæ of the points common to the locus and the x-axis.

If the degree of $f(x)$ is greater than 2, the computation necessary in order to draw the curve enables us to make a fair approximation to the roots without the figure. Nevertheless, a consideration of the figure often gives us important information concerning the roots.

This is illustrated in the proof of the following

Theorem. — *If a and b are real numbers and $f(a)$ and $f(b)$ have opposite signs, there is at least one real root of the equation $f(x)=0$ between a and b.*



If $f(a)$ and $f(b)$ have opposite signs, the locus of the equation $y=f(x)$ is on opposite sides of the x-axis at the points whose abscissæ are a and b respectively.

Now this locus is an unbroken curve and does not turn back on itself (see figure), since for each value of x there is only one value for y, and therefore it must cross the x-axis at least once between the point whose abscissa is a and the point whose abscissa is b. This proves the theorem.

This argument rests upon the assumption that the locus of the equation $y=f(x)$ is an unbroken curve. The assumption can be proved, but the proof is too difficult to be given here.

EXERCISES

Find the real roots of the following equations graphically:

1. $x^2+3x+2=0$.
2. $x^2-4x+4=0$.
3. $x^3+2x^2+2x+1=0$.
4. $x^3-6x^2-7x-6=0$.
5. $x^3+4x^2+2x-3=0$.
6. $x^3+4x^2+5x+2=0$.
7. $2x^3-3x^2-12x+1=0$.
8. $x^3-2x^2-2x-3=0$.

147. The fundamental theorem of algebra. — *Every equation of the form* $f(x) = 0$ *has at least one root.*

This theorem, which is usually referred to as the fundamental theorem of algebra, was first proved by Karl Friedrich Gauss, a German mathematician, in 1799. The proof is too difficult to be given here.*

148. Theorem. — *Any polynomial* $f(x)$ *of degree* n *has linear factors of the form* $x - r$, *where* r *is a real or a complex number.*

By the fundamental theorem the equation

$$f(x) = 0$$

has at least one root, say r_1. Then by the factor theorem $f(x)$ is divisible by $x - r_1$, and if $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n$, we have,

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \equiv (x - r_1)(a_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}).$$

Again by the fundamental theorem the equation

$$a_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1} = 0$$

has at least one root, say r_2.

Then using the factor theorem again we see that

$$a_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1} \equiv (x - r_2)(a_0x^{n-2} + c_1x^{n-3} + \cdots + c_{n-2}),$$

and that therefore

$$f(x) \equiv (x - r_1)(x - r_2)(a_0x^{n-2} + c_1x^{n-3} + \cdots + c_{n-2}).$$

By continuing this argument we see that we shall finally get $f(x)$ expressed as the product of n linear factors. Thus

$$f(x) \equiv a_0(x - r_1)(x - r_2) \cdots (x - r_n). \tag{1}$$

This shows that $r_1, r_2, \cdots, r_n$ are roots of the equation

$$f(x) = 0.$$

* The student who is interested will find a proof in Fine's *College Algebra*, p. 588.

These numbers may not all be distinct. If

$$f(x) = a_0(x - r_1)^{n_1}(x - r_2)^{n_2} \cdots (x - r_k)^{n_k},$$

we agree to say that r_i is a root of *multiplicity* n_i and to count it n_i times. Thus r_1 is a root of multiplicity n_1 and we count it n_1 times; r_2 is a root of multiplicity n_2 and we count it n_2 times; and so on. We have then the following

Corollary. — *Every equation of degree n has at least n roots.*

149. Theorem. — *No equation of degree n has more than n distinct roots.*

For suppose that the equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \text{ where } a_0 \neq 0.$$

The preceding theorem shows that the left member can be written in the form

$$a_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

If now r were a root of this equation and distinct from r_1, r_2, ..., r_n, we should have

$$a_0(r - r_1)(r - r_2) \cdots (r - r_n) = 0.$$

But the first factor, a_0, of the left member of this relation is by hypothesis not zero. Moreover none of the other factors is zero, since, by hypothesis r is different from r_1, r_2, ..., r_n. But no product can be zero unless at least one of the factors is zero (10, Chapter I). Hence the supposition that r can be a root of $f(x) = 0$ is wrong, and the theorem is proved.

Corollary. — *Every equation of degree n has exactly n roots in the sense explained in* § 148.

Corollary. — *If two polynomials in one variable of degrees not greater than n are equal to each other for more than n values of the variable the coefficients of like powers of the variable in the two polynomials are equal, and conversely.*

If $a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n$

for more than n values of x, then the relation

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0$$

is satisfied by more than n values of x. If some of the coefficients besides the last one in the left member of this relation were not zero, we should have an equation of degree not greater than n with more than n roots. But the theorem shows that this is impossible. Hence all these coefficients are zero, and therefore the last coefficient is also zero. That is,

$$\begin{aligned} &a_0 - b_0 = 0, \quad \text{or} \quad a_0 = b_0; \\ &a_1 - b_1 = 0, \quad \text{or} \quad a_1 = b_1; \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &a_n - b_n = 0, \quad \text{or} \quad a_n = b_n. \end{aligned}$$

The converse is obvious.

150. Relations between roots and coefficients. — If $r_1, r_2, \cdots, r_n$ are the roots of the equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad (a_0 = 1)$$

then (§ 148)

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \equiv (x - r_1)(x - r_2) \cdots (x - r_n).$$

If we perform the indicated multiplications in the right member of this identity, we shall get a polynomial which is equal to the left member of the identity for all values of x. Hence the coefficients of corresponding powers of x in the two polynomials must be equal (§ 149), and we have

$$a_1 = -r_1 - r_2 - \cdots - r_n. \tag{1}$$

$$a_2 = r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots + r_{n-1}r_n. \tag{2}$$

$$a_3 = -r_1r_2r_3 - r_1r_2r_4 - \cdots - r_{n-2}r_{n-1}r_n. \tag{3}$$

$$\cdots\cdots\cdots\cdots$$

$$a_n = (-1)^n r_1r_2 \cdots r_n. \tag{n}$$

The second member of (1) is the sum of all the roots with their signs changed. The second member of (2) is the sum of the products of the roots with their signs changed taken

two at a time. The second member of (3) is the sum of the products of the roots with their signs changed taken three at a time. And so on.

It should be noted that these relations hold only when the coefficient of the highest power of x is 1. If it is not 1, we must divide each member of the equation by it before applying these relations.

151. Imaginary roots. — Although we are interested here only in the real roots of an equation, the following theorem concerning the appearance of imaginary roots will be useful.

Theorem. — *If $c + id$ is a root of an equation with real coefficients, $c - id$ is also a root.*

Let the equation be

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where a_0, a_1, $\cdots$, a_n are real numbers, and let $f(x)$ be a symbol for the left member of this equation.

The product of the two factors $x - (c + id)$ and $x - (c - id)$ is $x^2 - 2cx + c^2 + d^2$. If now we divide $f(x)$ by this product, we shall get a quotient which we can represent by the symbol $Q(x)$, and a remainder which must be of degree less than 2. Then

$$f(x) \equiv Q(x)(x^2 - 2cx + c^2 + d^2) + rx + r',$$

where r and r' are certain real constants.*

Now this relation holds for all values of x, and therefore for $x = c + id$. But by hypothesis $f(c + id) = 0$. Therefore

$$0 = Q(c + id)(c^2 + 2icd - d^2 - 2c^2 - 2icd + c^2 + d^2) + r(c + id) + r',$$

or $$0 = Q(c + id) \cdot 0 + rc + ird + r'.$$

Hence, $$rc + r' + ird = 0.$$

It follows from this by § 123, that $rd = 0$ and $rc + r' = 0$. Hence either $r = 0$ or $d = 0$.

* If the coefficients in $f(x)$ were not all real, we could not be sure that r and r' would be real.

But if $d = 0$ the root $c + id$ would be real. Therefore $r = 0$, and then $r' = 0$.

Hence, $$f(x) \equiv Q(x)(x^2 - 2cx + c^2 + d^2),$$

and therefore

$$f(c - id) \equiv Q(c - id)(c^2 - 2icd - d^2 - 2c^2 + 2icd + c^2 + d^2)$$
$$= Q(c - id) \cdot 0 = 0.$$

This means that $c - id$ is a root of the equation $f(x) = 0$.

Corollary. — *Every equation of odd degree with real coefficients has at least one real root.*

152. Transformations of equations. — In solving an equation $f(x) = 0$ it is frequently necessary to transform it into another equation whose roots bear a given relation to the roots of $f(x) = 0$. In this book we shall make use of three transformations.

153. I. *To transform an equation $f(x) = 0$ into one whose roots are those of $f(x) = 0$ with their signs changed.*

If we write $f(x)$ in the form

$$f(x) = a_0(x - r_1)^{n_1}(x - r_2)^{n_2} \cdots (x - r_k)^{n_k},$$

we see that

$$f(-x) = a_0(-x - r_1)^{n_1}(-x - r_2)^{n_2} \cdots (-x - r_k)^{n_k},$$

and that therefore the roots of $f(-x) = 0$ are $-r_1, -r_2, \cdots, -r_k$.

Moreover each of these roots has the same multiplicity as the corresponding root of $f(x) = 0$.

Hence every root of $f(x) = 0$ with its sign changed is a root of $f(-x) = 0$.

If the given equation is

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

the transformed equation is

$$a_0(-x)^n + a_1(-x)^{n-1} + \cdots + a_n = 0.$$

This can be simplified into

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n = 0.$$

Whether the sign before a_n is $+$ or $-$ depends upon whether n is even or odd. Hence we have written this term $(-1)^n a_n$, since this equals a_n when n is even and $-a_n$ when n is odd.

We have therefore the following

Rule. — *In order to form an equation whose roots are those of* $f(x) = 0$ *of degree* n, *with their signs changed, change the signs of the alternate coefficients of* $f(x) = 0$, *beginning with the coefficient of* x^{n-1}. *If any power of* x *is lacking, its coefficient must be considered as zero.*

Example. — Form an equation whose roots are those of

$$2x^4 + 5x^3 - 6x^2 + 3 = 0$$

with their signs changed.

The required equation is $2x^4 - 5x^3 - 6x^2 + 3 = 0$.

154. II. *To transform an equation* $f(x) = 0$ *into one whose roots are those of* $f(x) = 0$ *each multiplied by a constant* m.

If we write $f(x)$ in the form

$$f(x) = a_0(x - r_1)^{n_1}(x - r_2)^{n_2} \cdots (x - r_k)^{n_k},$$

we see that

$$f\left(\frac{x}{m}\right) = a_0\left(\frac{x}{m} - r_1\right)^{n_1}\left(\frac{x}{m} - r_2\right)^{n_2} \cdots \left(\frac{x}{m} - r_k\right)^{n_k},$$

and that therefore the roots of

$$f\left(\frac{x}{m}\right) = 0$$

are mr_1, mr_2, $\cdots$, mr_k.

Moreover each of these roots has the same multiplicity as the corresponding root of $f(x) = 0$.

Hence the product of m and any root of $f(x) = 0$ is a root of

$$f\left(\frac{x}{m}\right) = 0.$$

If the given equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

the transformed equation is

$$a_0\left(\frac{x}{m}\right)^n + a_1\left(\frac{x}{m}\right)^{n-1} + a_2\left(\frac{x}{m}\right)^{n-2} + \cdots + a_n = 0.$$

When cleared of fractions this becomes

$$a_0x^n + ma_1x^{n-1} + m^2a_2x^{n-2} + \cdots + m^na_n = 0.$$

We have therefore the following

Rule. — *In order to form an equation whose roots are those of the equation $f(x) = 0$ of degree n, each multiplied by the constant m, multiply the successive coefficients beginning with the coefficient of x^{n-1} by m, $m^2, \cdots, m^n$ respectively. If any power of x is lacking, its coefficient must be considered as zero.*

When $m = -1$ this transformation is the same as Transformation I.

Example I. — Form an equation whose roots are the roots of

$$2x^5 + 5x^3 - 6x^2 + 3 = 0,$$

each multiplied by 3.

The required equation is

$$2x^5 + 3^2 \cdot 5x^3 - 3^3 \cdot 6x^2 + 3^5 \cdot 3 = 0,$$

or

$$2x^5 + 45x^3 - 162x^2 + 729 = 0.$$

Example II. — Form an equation whose roots are the roots of

$$x^4 - 3x^2 - 5x + 1 = 0,$$

each divided by 2.

Here $m = \frac{1}{2}$. Hence the new equation is

$$x^4 - (\tfrac{1}{2})^2 \cdot 3x^2 - (\tfrac{1}{2})^3 \cdot 5x + (\tfrac{1}{2})^4 = 0,$$

or

$$16x^4 - 12x^2 - 10x + 1 = 0.$$

Example III. — Transform the equation

$$2x^3 - 5x^2 - x + 3 = 0$$

into one whose roots are the roots of this equation each multiplied by m. Then find the least positive value of m for which the new equation shall

have its coefficients all integers when the coefficient of the highest power of x is 1.

If we make the coefficient of x^3 in the given equation equal to 1, we get

$$x^3 - \tfrac{5}{2}x^2 - \tfrac{1}{2}x + \tfrac{3}{2} = 0.$$

Now
$$x^3 - \frac{5m}{2}x^2 - \frac{m^2}{2}x + \frac{3m^3}{2} = 0$$

is the equation whose roots are the roots of this given equation each multiplied by m.

We wish to give to m the least positive value that will make the coefficients of this last equation integers. We see by inspection that this value is 2. The resulting equation is

$$x^3 - 5x^2 - 2x + 12 = 0.$$

Its roots are equal to those of the original equation each multiplied by 2.

The least positive value of m called for in problems like this one is either the coefficient of the highest power of x in the original equation or a divisor of this coefficient.

This example illustrates an application of Transformation II that we shall have frequent occasion to make.

155. III. *To transform an equation $f(x) = 0$ into one whose roots are those of $f(x) = 0$ each diminished by a constant h.*

If we write $f(x)$ in the form

$$f(x) = a_0(x - r_1)^{n_1}(x - r_2)^{n_2} \cdots (x - r_k)^{n_k},$$

we can see that

$$f(x + h) = a_0(x + h - r_1)^{n_1}(x + h - r_2)^{n_2} \cdots (x + h - r_k)^{n_k}$$

and that therefore the roots of

$$f(x + h) = 0$$

are
$$r_1 - h,\ r_2 - h,\ \cdots,\ r_k - h.$$

Moreover each of these roots has the same multiplicity as the corresponding root of $f(x) = 0$. Hence any root of $f(x) = 0$ diminished by h is a root of $f(x + h) = 0$.

If the given equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \qquad (1)$$

the transformed equation is

$$a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \cdots + a_n = 0. \quad (2)$$

This can be simplified by expanding the powers of the binomials and collecting the resulting terms. The final equation will be in the form

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_n = 0. \quad (3)$$

In most cases the labor involved in determining the values of A_0, A_1, A_2, $\cdots$, A_n in this way is very great, and we therefore look for a shorter way of doing this.

If in equation (2) we put $x - h$ in place of x, we get back to equation (1). Hence, since the left member of (3) is merely another form of the left member of (2),

$$\begin{aligned} &a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \\ &\quad \equiv A_0(x-h)^n + A_1(x-h)^{n-1} + \cdots A_{n-1}(x-h) + A_n \\ &\quad \equiv [A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + \cdots + A_{n-1}](x-h) + A_n. \end{aligned}$$

This shows that if we divide the left member of (1) by $x - h$, the remainder is A_n. If we divide the quotient by $x - h$, the remainder is A_{n-1}, and if we divide this second quotient by $x - h$, the remainder is A_{n-2}. By working back in this way we can get all the coefficients in the left member of (3), A_0 being the last quotient, and A_1 the last remainder. We have, therefore, the following

Rule. — *In order to form the equation whose roots are the roots of $f(x) = 0$ each diminished by h, divide $f(x)$ and each successive resulting quotient by $x - h$ until a constant quotient is obtained.*

The first remainder is the constant term and each successive remainder is the coefficient of the next higher power of x, in the desired equation, the last quotient being the coefficient of the highest power of x.

The advantage of computing these coefficients in this way lies in the fact that the work can be shortened by the use of synthetic division.

Example. — Form an equation whose roots are the roots of

$$2x^4 - 5x^3 + 7x - 4 = 0$$

each diminished by 2.

$$\begin{array}{rrrrr|l} 2 - & 5 + & 0 + & 7 - & 4 & 2 \\ & 4 - & 2 - & 4 + & 6 & \\ \hline 2 - & 1 - & 2 + & 3 & +2 & \\ & + 4 + & 6 + & 8 & & \\ \hline 2 + & 3 + & 4 & +11 & & \\ & + 4 + & 14 & & & \\ \hline 2 + & 7 & +18 & & & \\ & + 4 & & & & \\ \hline 2 & +11 & & & & \end{array}$$

The desired equation is

$$2x^4 + 11x^3 + 18x^2 + 11x + 2 = 0.$$

156. Graphical interpretation of Transformation III. — If we represent the left member of equation (3) of the preceding article by $F(x)$, the value of y in the equation $y = F(x)$ is the same for $x = a - h$ as it is in the equation $y = f(x)$ for $x = a$. Hence $y = F(x)$ is the equation of the locus of $y = f(x)$ when the y-axis is moved h units to the right of its original position. It is to be understood that if h is negative, the new y-axis is to the left of the old one.

For example, to move this axis -2 units to the right is to move it 2 units to the left.

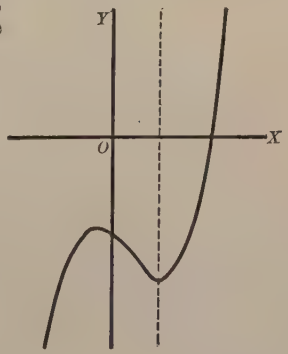
Consider the equation $x^3 - x^2 - x - 2 = 0$, and form the equation whose roots are the roots of this, each diminished by 1.

$$\begin{array}{rrrr|l} 1 - & 1 - & 1 - & 2 & 1 \\ & + 1 + & 0 - & 1 & \\ \hline 1 + & 0 - & 1 & -3 & \\ & + 1 + & 1 & & \\ \hline 1 + & 1 & +0 & & \\ & + 1 & & & \\ \hline 1 & +2 & & & \end{array}$$

The new equation is $x^3 + 2x^2 - 3 = 0$.

The locus of the equation

$$y = x^3 - x^2 - x - 2$$



is given in the figure and the locus of the equation

$$y = x^3 + 2x^2 - 3$$

is the same curve provided we move the y-axis one unit to the right of its original position. The broken line is the new y-axis.

EXERCISES

1. Does the proof of the second corollary of the theorem of § 149 apply when the two polynomials are not of the same degree?

Form equations whose roots are the roots of the following equations with their signs changed:

2. $x^3 - 4x^2 - 5x + 1 = 0$.
3. $6x^5 + 7x^3 + 2x^2 - x - 8 = 0$.
4. $12x^4 + 11x^2 + 2 = 0$.
5. $2x^4 + 3x^2 - 4x + 1 = 0$.
6. $x^3 = 3x^2 - 2x + 5$.
7. $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$.
8. $x^3 + 1 = 0$.
9. $8x^4 - 2x^3 + 10x^2 + 5x + 7 = 0$.
10. $-4x^3 - 5x^2 - 3x + 2 = 0$.
11. $5x^3 - 8x^2 - x + 2 = 0$.
12. $3x^4 + 7x^3 + 5x^2 + 2x + 8 = 0$.
13. $-2x^4 - 3x^3 + 1 = 0$.

Form equations whose roots are the roots of the following equations each multiplied by the numbers placed opposite the respective equations:

14. $x^4 + 2x^3 + 3x^2 + 4x + 5 = 0$, 2.
15. $5x^3 - x^2 + 4 = 0$, 3.
16. $2x^4 + 10x^3 - 7x^2 + x + 4 = 0$, $\frac{1}{2}$.
17. $5x^4 - x^3 + 7x^2 + 3x - 4 = 0$, 5.
18. $x^3 + x^2 + x + 1 = 0$, -2.
19. $-3x^4 + 10x^3 + 6x^2 - x - 4 = 0$, $\frac{1}{3}$.

Form equations whose roots are the roots of the following equations each multiplied by m. Then find the least positive

value of m for which the coefficients of the resulting equation in each case are integers when the coefficient of the highest power of x is 1.

20. $6x^3+5x^2+x-2=0$.

21. $x^4-2x^3+7x^2+\frac{x}{2}+3=0$.

22. $2x^4+4x=1$.

23. $x^3+\frac{1}{27}=0$.

24. $9x^4-4x^3-6x^2+9x+18=0$.

25. $5x^3+2x+1=0$.

26. $5x^3+2x^2+x-3=0$.

27. $2x^4-7x^3-10x^2+x-8=0$.

28. $3x^5+x^4-2x^3-4x^2-5x-2=0$.

29. $3x^4+x^2+x+5=0$.

30. $-2x^3+4x^2+6x+8=0$.

31. What are the roots of $x^3+2x^2-2x+3=0$, it being given that 3 is a root of $x^3-2x^2-2x-3=0$?

32. What are the roots of $x^3+4x^2-7x-10=0$, it being given that -2 is a root of $x^3-4x^2-7x+10=0$?

Form equations whose roots are the roots of the following equations, each diminished by the number placed opposite the respective equation. Then plot the loci of the equations obtained by putting the left members of the original equations equal to y, and show the effect on the y-axis of each transformation.

33. $x^2-5x+6=0$, 1.

34. $x^3-3x^2-x+3=0$, 2.

35. $x^3-1=0$, 1.

36. $x^3+6x^2+9x+20=0$, 3.

37. $x^2+6x=8+x^2$, 2.

38. $x^2-5x+6=0$, -1.

39. $x^2+3x+2=0$, 2.

40. $x^2-2x+1=0$, 1.

41. $x^3-3x^2+3x-1=0$, 1.

42. $x^3-6x^2-11x-6=0$, 2.

43. What relations do the loci of the equations $y=-f(x)$ and $y=f(-x)$ bear to the locus of $y=f(x)$? Apply your answer to the cases in which $f(x)$ is each of the polynomials in Exs. 33–37 in turn.

157. Continuations and variations in sign. — The presence of two consecutive terms with like signs in a polynomial with real coefficients is called a **continuation** in sign, and the presence of two consecutive terms with unlike signs is called a **variation** in sign.

Thus, in $3x^5 - 7x^4 - x^3 + x + 3$ there are two continuations and two variations in sign. It is not necessary to take into account the missing terms.

Consider the product arising from the multiplication of

$$\begin{array}{l} 3x^4 - 2x^3 + 5x^2 + 3x + 2 \text{ by } x - 2. \\ 3x^4 - 2x^3 + 5x^2 + \ 3x + 2 \\ \qquad\qquad\qquad\qquad\quad x - 2 \\ \hline 3x^5 - 2x^4 + 5x^3 + \ 3x^2 + 2x \\ \qquad - 6x^4 + 4x^3 - 10x^2 - 6x - 4 \\ \hline 3x^5 - 8x^4 + 9x^3 - \ 7x^2 - 4x - 4 \end{array}$$

Here there are two variations in sign in the original polynomial, and three variations in the product. This illustrates what happens in general, as is shown by the following

Theorem. — *If $f(x)$ is a polynomial with real coefficients and r is a positive number, the number of variations in sign in $(x - r)f(x)$ is at least one more than the number of variations in $f(x)$.*

The proof of this theorem is based on the following

Underlying principle. — *If, in a sequence of real numbers $c_1, c_2, \cdots, c_p$, the signs of c_1 and c_p are opposite, the sequence has at least one variation of sign.*

If $$f(x) = a_0x^n + a_1a^{n-1} + \cdots + a_{n-k}x^k,$$

where a_0 and a_{n-k} are different from 0, and k is greater than, or equal to, 0, then

$$(x - r)f(x) = A_0x^{n+1} + A_1x^n + \cdots + A_{n-k}x^{k+1} + A_{n-k+1}x^k,$$

where $$A_0 = a_0,\ A_{n-k+1} = -ra_{n-k},$$

and for all other coefficients,

$$A_p = a_p - ra_{p-1}.$$

In this last formula p can have any value from 1 to $n-k$ inclusive.

If any two consecutive coefficients of $f(x)$, as a_{p-1} and a_p, have opposite signs, A_p has the same sign as a_p. For the formula for A_p shows that it is positive when a_{p-1} is negative and a_p is positive, and that it is negative when a_{p-1} is positive and a_p is negative.

Suppose now that the last variation in sign in the sequence $a_0, a_1, \cdots, a_{n-k}$ occurs between a_{l-1} and a_l. Then since $A_0 = a_0$ there must be at least as many variations in sign in the sequence $A_0, A_1, \cdots, A_l$ as there are in the sequence $a_0, a_1, \cdots, a_l$, and therefore at least as many as there are in the sequence $a_0, a_1, \cdots, a_{n-k}$. But there must be at least one variation in sign from A_l to A_{n-k+1} since the former has the same sign as a_{n-k} and the latter has the opposite sign $(A_{n-k+1} = -ra_{n-k})$.

Hence there is at least one more variation in sign in $(x-r)f(x)$ than in $f(x)$.

158. Descartes's Rule of Signs. — *An equation $f(x)=0$ with real coefficients cannot have more positive roots than $f(x)$ has variations in sign, nor more negative roots than $f(-x)$ has variations in sign.*

I. Suppose that

$$r_1, r_2, \cdots, r_p \text{ are the positive roots of } f(x) = 0,$$

and that $$f(x) \equiv (x-r_1)(x-r_2)\cdots(x-r_p)Q(x).$$

The number of variations in sign in $(x-r_p)Q(x)$ exceeds the number of variations in $Q(x)$ by at least one. And similarly when we multiply by each of the $p-1$ remaining factors $(x-r_1), (x-r_2), \cdots, (x-r_{p-1})$ in turn we add at least one variation in sign to the product. Hence there must be at least p variations in sign in $f(x)$.

II. Since the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$ (§ 153), the number of these negative roots cannot exceed the number of variations in sign in $f(-x)$.

It should be observed that Descartes's rule of signs does not tell us how many positive or negative roots the equation $f(x)=0$ has; it merely tells us that the equation cannot have more than a certain number of each of these kinds of roots. It gives us no information about the number of zero roots. But these roots can readily be determined by inspection. In finding the other real roots it is best first to divide $f(x)$ by the highest power of x by which it is divisible. If this power is the ath, and

$$f(x)=x^a f_1(x),$$

then 0 is a root of $f(x)=0$ of multiplicity a and the non-zero roots of $f(x)=0$ are the roots of $f_1(x)=0$.

If $f(x)$ has just one variation in sign, it can be shown that the equation $f(x)=0$ has one positive root. We shall omit the proof of this statement.

EXERCISES

Use Descartes's rule of signs to obtain all the information you can about the number of positive and of negative roots of each of the following equations. Can you tell whether any of the equations have imaginary roots?

1. $x^4+3x^2+5x-1=0$.
2. $x^4+3x^2-5x+1=0$.
3. $x^4-3x^2+5x+1=0$.
4. $x^5-6=0$.
5. $2x^6+7x^4+x^2+5=0$.
6. $x^3+3x^2-2x-5=0$.
7. $4x^4+x^3+5x^2+3=0$.
8. $x^6-2=0$.
9. $x^3+2x^2+3x+4=0$.
10. $3x^3+16x^2+18x-20=0$.

11. The roots of the equation

$$6x^4+29x^3-54x^2-51x-10=0$$

are all real. How many of them are positive and how many are negative?

12. The equation $6x^4 - x^3 - x^2 - 75x - 25 = 0$

has an imaginary root. How many of its roots are negative?

13. Show that the equation

$$2x^6 - 5x^2 - x + 2 = 0$$

has at least two imaginary roots.

159. Theorem. — *Every rational root of the equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

with integral coefficients is an integer and a divisor of a_n.

Suppose that the given equation has a fractional root, and that this fraction in its lowest terms is $\frac{p}{q}$. This means that $q > 1$ and that p and q have no common divisor except 1. Then we have the relation

$$\frac{p^n}{q^n} + a_1\frac{p^{n-1}}{q^{n-1}} + a_2\frac{p^{n-2}}{q^{n-2}} + \cdots + a_n = 0.$$

Multiplying both members by q^{n-1} and transposing, we get

$$\frac{p^n}{q} = -(a_1p^{n-1} + a_2qp^{n-2} + \cdots + a_nq^{n-1}).$$

Now the right member of this relation is an integer, and therefore if $\frac{p}{q}$ were a root of the given equation, p^n would be divisible by q. But this is impossible, since p and q have no common divisor except 1 and $q > 1$. Hence every rational root of the equation must be an integer.

If r is an integral root of the equation, we have

$$r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_n = 0,$$

or

$$r^{n-1} + a_1r^{n-2} + a_2r^{n-3} + \cdots + a_{n-1} = -\frac{a_n}{r}.$$

Now the left member of this relation is an integer and therefore $\frac{a_n}{r}$ is an integer. This is equivalent to saying that a^n is divisible by r.

This completes the proof of the theorem.

160. We are now in a position to solve the main problem of this chapter; namely, to find the real roots of an equation with rational coefficients.

We consider first the problem of finding the rational roots of such an equation.

161. Rational roots. — If the equation is of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \tag{1}$$

where $a_0 \neq 0$, we first form a new equation whose roots are m times the roots of this equation, and find the least positive value of m for which this new equation has integral coefficients when the coefficient of its highest power of x is 1. Suppose that the new equation for this value of m is

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0 \tag{2}$$

in which the coefficients are integers. If equation (1) has a rational root, equation (2) has an integral root which is a divisor of b_n.

We therefore determine by synthetic division what positive and negative divisors of b_n are roots of equation (2). If we divide every such root by m, we shall get all the rational roots of equation (1).

When one root r of (2) has been found, divide the left member of (2), which we will call $F(x)$, by $x - r$.

$$F(x) \equiv (x - r)Q(x).$$

Every other root of (2) is a root of $Q(x)=0$, and these other roots can best be obtained from this last equation. We shall refer to it as the *depressed* equation since it is of lower degree than (1).

Sometimes the labor of finding these roots can be shortened by determining the maximum number of positive and negative roots by means of Descartes's rule of signs.

Example. — Find the rational roots of

$$9x^4 + 12x^3 + 10x^2 + x - 2 = 0. \tag{1}$$

The equation whose roots are m times the roots of this can be written in the form

$$x^4 + \frac{12m}{9}x^3 + \frac{10m^2}{9}x^2 + \frac{m^3x}{9} - \frac{2m^4}{9} = 0.$$

The least positive value of m that will make these coefficients integers is 3. For this value of m the equation is

$$x^4 + 4x^3 + 10x^2 + 3x - 18 = 0. \tag{2}$$

We know from § 158 that this equation has just one positive root and from § 159 that this root is a divisor of 18, if it is rational. Now the positive divisors of 18 are 1, 2, 3, 6, 9, 18; and we find by synthetic division that 1 is a root.

$$\begin{array}{r|l} 1 + 4 + 10 + 3 - 18 & 1 \\ \quad + 1 + 5 + 15 + 18 & \\ \hline 1 + 5 + 15 + 18 + 0 & \end{array}$$

All the other roots are roots of the depressed equation

$$x^3 + 5x^2 + 15x + 18 = 0.$$

This equation has no positive root, and any negative root it may have is a negative divisor of 18.

Try -1.

$$\begin{array}{r|l} 1 + 5 + 15 + 18 & -1 \\ \quad - 1 - 4 - 11 & \\ \hline 1 + 4 + 11 + 7 & \end{array}$$

Hence -1 is not a root.

Try -2.

$$\begin{array}{r|l} 1 + 5 + 15 + 18 & -2 \\ \quad - 2 - 6 - 18 & \\ \hline 1 + 3 + 9 + 0 & \end{array}$$

Hence -2 is a root, and the other roots are roots of the depressed equation

$$x^2 + 3x + 9 = 0,$$

and therefore imaginary.

Since 1 and -2 are all the rational roots of (2), $\frac{1}{3}$ and $-\frac{2}{3}$ are all the rational roots of (1), since the roots of (2) are the roots of (1) each multiplied by 3.

It is not necessary to perform any extra work to get the successive depressed equations since the left member of each one is obtained incidentally in finding a root of the preceding equation.

EXERCISES

Find all the rational roots of each of the following equations:

1. $x^3+x^2+x+1=0$.
2. $2x^3+x^2+x-1=0$.
3. $x^3+5x^2-2x+2=0$.
4. $x^4-5x^3+10x-7x-2=0$.
5. $x^3+x^2-16x+20=0$.
6. $2x^3+9x^2+11x+3=0$.
7. $x^4+3x^3=x^2$.
8. $x^4+x^3+x^2+x+1=0$.
9. $x^3-3x^2+5x+4=0$.
10. $8x^3-4x^2-2x+1=0$.
11. $6x^3-2x^2+4x-1=0$.
12. $4x^3-16x^2-9x+36=0$.
13. $4x^3+8x^2-x-2=0$.
14. $6x^4+2x^2+5=0$.
15. $4x^3-5x-6=0$.
16. $3x^3+16x^2+18x-20=0$.
17. $2x^3+3x^2+5x+2=0$.
18. $x^4+3x^2+2=0$.
19. $x^4-3x^3+3x^2-3x+2=0$.
20. $x^5-32=0$.

162. Irrational roots. — The first step in finding an irrational real root of $f(x)=0$ is to find two consecutive integers between which this root lies. This can be done by making use of the theorem of § 146.

The following method for computing approximately the irrational real roots of an equation is based upon this theorem. It is known as **Horner's Method**, from the name of its inventor.

163. Positive irrational roots. — The essential features of this method can best be explained by means of an example.

Example. — Find the irrational real roots of the equation

$$x^3+3x^2-2x-5=0. \tag{1}$$

1. The left member, which we represent by $f(x)$, has one variation of sign and therefore the equation has just one positive root.

2. In order to determine the location of this positive root, we substitute for x in $f(x)$ successive integral values of x be-

ginning with 0. We see directly that $f(0) = -5$ and indirectly by means of synthetic division that $f(1) = -3$ and $f(2) = 11$.

$$\begin{array}{r|l} 1+3-2-5 & 1 \\ \quad +1+4+2 & \\ \hline 1+4+2-3 & \end{array} \qquad \begin{array}{r|l} 1+3-\ 2-\ 5 & 2 \\ \quad +2+10+16 & \\ \hline 1+5+\ 8+11 & \end{array}$$

Hence by the theorem of § 146, the equation has a root between 1 and 2.

3. Form a new equation whose roots are equal to the roots of $f(x) = 0$ each diminished by 1.

$$\begin{array}{r|l} 1+3\ -2\ -5 & 1 \\ \quad +1\ +4\ +2 & \\ \hline 1+4\ +2 & -3 \\ \quad +1\ +5 & \\ \hline 1+5 & +7 \\ \quad +1 & \\ \hline 1+6 & \end{array}$$

This new equation is

$$x^3 + 6x^2 + 7x - 3 = 0, \qquad (2)$$

and it has a root between 0 and 1.

4. For $x = 0$, .1, .2, .3, and .4 the left member of this equation equals respectively -3, -2.239, -1.352, $-.333$, and .824, as may be seen from the following computations:

$$\begin{array}{r|l} 1+6\ +7\ \ -3 & .1 \\ \quad +\ .1+\ .61+\ .761 & \\ \hline 1+6.1+7.61-2.239 & \end{array} \qquad \begin{array}{r|l} 1+6\ +7\ \ -3 & .2 \\ \quad +\ .2+1.24+1.648 & \\ \hline 1+6.2+8.24-1.352 & \end{array}$$

$$\begin{array}{r|l} 1+6\ +7\ \ -3 & .3 \\ \quad +\ .3+1.89+2.667 & \\ \hline 1+6.3+8.89-\ .333 & \end{array} \qquad \begin{array}{r|l} 1+6\ +7\ \ -3 & .4 \\ \quad +\ .4+2.56+3.824 & \\ \hline 1+6.4+9.56+\ .824 & \end{array}$$

Hence equation (2) has a root between .3 and .4.

5. Form a new equation whose roots are the roots of (2) each diminished by .3.

$$\begin{array}{l|l} 1+6\ \ +7\ \ -3 & .3 \\ \quad .3+1.89+2.667 & \\ \hline 1+6.3+8.89 & -.333 \\ \quad +\ .3+1.98 & \\ \hline 1+6.6 & +10.87 \\ \quad\ .3 & \\ \hline 1+6.9 & \end{array}$$

The new equation is

$$x^3+6.9\,x^2+10.87\,x-.333=0 \tag{3}$$

and it has a root between 0 and .1.

6. We find by trial that this root lies between .03 and .04.

$$\begin{array}{l|l} 1+6.9\ \ +10.87\ \ -.333 & .03 \\ \quad .03+\ \ .2079+.332337 & \\ \hline 1+6.93+11.0779-.000663 & \end{array}$$

$$\begin{array}{l|l} 1+6.9\ \ +10.87\ \ -.333 & .04 \\ \quad +\ .04+\ \ .2776+.445904 & \\ \hline 1+6.94+11.1476+.112904 & \end{array}$$

7. Form a new equation whose roots are the roots of (3) each diminished by .03.

$$\begin{array}{l|l} 1+6.9\ \ +10.87\ \ -.333 & .03 \\ \quad .03+\ \ .2079+.332337 & \\ \hline 1+6.93+11.0779 & -.000663 \\ \quad +\ .03+\ \ .2088 & \\ \hline 1+6.96 & +11.2867 \\ \quad +\ .03 & \\ \hline 1+6.99 & \end{array}$$

The new equation is

$$x^3+6.99\,x^2+11.2867\,x-.000663=0 \tag{4}$$

and it has a root between 0 and .01.

8. We find by trial that this root lies between 0 and .001.

$$\begin{array}{l|l} 1+6.99\ \ +11.2867\ \ -.000663 & .001 \\ \quad +\ .001+\ \ .006991+.011293691 & \\ \hline 1+6.991+11.293691+.010630691 & \end{array}$$

Obviously the left member of (4) is negative for $x=0$.

9. Since each root of (4) is .03 less than a root of (3), the latter has a root between .03 and .031.

Since each root of (3) is .3 less than a root of (2), the latter has a root between .33 and .331.

Since each root of (2) is 1 less than a root of (1), the latter has a root between 1.33 and 1.331.

Hence we have found a root of the original equation correct to two decimal places. If the digit in the third decimal place had been 5 or larger, the root to two decimal places would have been 1.34. In general, in order to get a root to r decimal places, it is necessary to determine whether the digit in the $(r+1)$th place is less than 5, or not.

In practice the body of this work can be compactly arranged as follows:

$1+3$	-2	-5	$\lfloor 1$
$+1$	$+4$	$+2$	
$1+4$	$+2$	-3	
$+1$	$+5$		
$1+5$	$+7$		
$+1$			
$1+6$	$+7$	-3	$\lfloor .3$
$+.3$	$+1.89$	$+2.667$	
$1+6.3$	$+8.89$	$-.333$	
$+.3$	$+1.98$		
$1+6.6$	$+10.87$		
$+.3$			
$1+6.9$	$+10.87$	$-.333$	$\lfloor .03$
$+.03$	$+.2079$	$+.332337$	
$1+6.93$	$+11.0779$	$-.000663$	
$+.03$	$.2088$		
$1+6.96$	$+11.2867$		
$.03$			
$1+6.99$	$+11.2867$	$-.000663$	

164. By an obvious continuation of this method we can compute this root to any number of decimal places.

The higher powers of a number between 0 and 1 are less than the number itself. Hence if an equation is known to have a root between 0 and 1, the terms containing the higher

powers of the unknown will be relatively unimportant, and the root can be determined approximately by neglecting these higher powers and solving the resulting equation of the first degree. The nearer the root is to zero, the closer this approximation will be. Applying this principle to (4) we see that .00006 is an approximate value of one of its roots, and this suggests that this root lies between 0 and .001, as was stated. But it must be kept clearly in mind that this is only a suggestion, and that the suggestion must be tested.

In a similar way we get the suggestions that .03 and .4 are approximate roots of (3) and (2) respectively.

These considerations enable us to do away with many synthetic divisions that otherwise would be necessary.

If in our efforts to locate a root between two consecutive integers we divide $f(x)$ by $x - r$, where r is positive, and find that the remainder and all the coefficients of the quotient are positive, we may conclude that r is greater than any real root of the equation $f(x) = 0$. For if we divide $f(x)$ by $x - r_1$, where $r_1 > r$, all these coefficients after the first one, as well as the remainder, will be greater than they were and therefore the remainder will not be zero. The value of this observation may be seen in step 3 of the example worked above. The suggestion just described tells us that the root we are looking for is approximately .4, and we accordingly divide the left member of equation (2) by $x - .4$.

$$\begin{array}{r|l} 1 + 6 \quad + 7 \quad - 3 & .4 \\ \quad + \quad .4 + 2.56 + 3.824 & \\ \hline 1 + 6.4 + 9.56 + \quad .824 & \end{array}$$

Since all these sums are positive, we know that the root cannot lie between .4 and .5, and we accordingly divide by $x - .3$ and note the sign of the remainder.

There are other considerations by means of which this process can be shortened, but we shall not take them up here.

When one irrational root has been found to the required degree of approximation, an entirely new start must be made in finding the next one. On the other hand, when a rational root has been found, the equation can be depressed (§ 161) and the finding of the remaining real roots, both rational and irrational, is thereby made correspondingly easier.

165. Negative irrational roots. — In order to find the negative irrational roots of $f(x)=0$, find the positive irrational roots of $f(-x)=0$. These with their signs changed are the roots sought.

166. Summary. — In order to find all the real roots of an equation $f(x)=0$, in which $f(x)$ is a polynomial with rational numerical coefficients, proceed as follows:

1. *Find all the rational roots by the method described in* § 161. *When each rational root has been found depress the corresponding equation.*

2. *See if the last depressed equation after all the rational roots have been found has any positive roots, and determine by synthetic division two consecutive integers between which such a root lies.*

3. *Form a new equation whose roots are the roots of this equation each diminished by the smaller of these two integers.*

4. *The resulting equation has a root between* 0 *and* 1. *Find by synthetic division the two consecutive tenths between which the root lies.*

5. *Form a new equation whose roots are the roots of this equation each diminished by the smaller of these tenths.*

6. *The resulting equation has a root between* 0 *and* .1. *Find by synthetic division two consecutive hundredths between which this root lies.*

7. *Form a new equation whose roots are the roots of this equation each diminished by the smaller of these hundredths.*

8. *If the root is required to r decimal places, continue this process until* $r+1$ *decimal places have been determined.*

9. *Add up the amounts by which the roots of the successive equations have been diminished. This sum, with the figure in the rth decimal place increased by* 1 *in case the figure in the* $(r+1)$*th place is* 5 *or more, is the root sought to the required degree of approximation.*

10. *If there are other positive irrational roots, find each of them in the same way.*

11. *In order to find the negative irrational roots, find the positive irrational roots of* $f(-x)=0$ *and change the sign of each one.*

12. *Make use of all the information that is obtainable from Descartes's rule of signs as to the number of positive and of negative roots.*

If in finding the irrational roots of $f(x)=0$ by Horner's Method we have any reason to suspect the existence of two roots between a and $a+1$, we should examine the sign of $f(x)$ for $x=a$, $x=a+\frac{1}{2}$, and $x=a+1$; and it may be necessary to examine the sign of $f(x)$ for values of x still closer together.

EXERCISES AND PROBLEMS

Find the values of the real roots of the following equations correct to two decimal places:

1. $2x^3+4x^2-10x+3=0$.
2. $5x^3-3x^2-6x+3=0$.
3. $x^4+x^3-7x^2-8x+20=0$.
4. $2x^4-x^3-x^2-x-3=0$.
5. $4x^3=3x-\frac{3}{4}$.
6. $2x^3-3x^2+\frac{1}{4}=0$.
7. $8x^4+8x^2-\frac{15}{2}=0$.
8. $8x^4=8x^2-\frac{3}{2}$.
9. $x^3-2x-2=0$.
10. $x^4-4x-2=0$.

11. How deep will a cork sphere 4 inches in diameter sink in water, the specific gravity of the cork being .2?

HINT. — The volume of a spherical segment of one base is given by the formula

$$v=\tfrac{1}{2}\pi x r_1^2+\tfrac{1}{6}\pi x^3,$$

where x is the altitude of the segment and r_1 is the radius of its base.

12. How deep will a sphere of pine of specific gravity $\frac{1}{2}$ sink in water?

13. If the specific gravity of ice is .9, how much of a sphere of ice 2 feet in diameter would protrude above the water in which the ice is floating?

14. The volume of a box $10 \times 12 \times 15$ inches is to be increased 50 cubic inches by adding the same amount to each dimension. What should this amount be?

15. How thick should a hollow spherical shell be whose inner radius is 3 inches in order to contain 40 cubic inches?

16. If a is the cosine of an angle and x is the cosine of one third of this angle, then $4x^3 = 3x + a$.

What is the cosine of an angle of 20°?

HINT. — The cosine of 60° is $\frac{1}{2}$.

17. A house may be bought for \$3400 cash, or in annual installments of \$1000 each, payable 1, 2, 3, and 4 years from date. What is the annual interest rate implied in this offer?

HINT. — The amount of \$3400 for 4 years should equal the sum of the amount of the first payment for 3 years, the amount of the second payment for 2 years, the amount of the third one for 1 year, and the fourth payment.

Hence, if x is the rate of interest,

$$3400(1+x)^4 - 1000(1+x)^3 - 1000(1+x)^2 - 1000(1+x) - 1000 = 0,$$

or, $$17(1+x)^4 - 5(1+x)^3 - 5(1+x)^2 - 5(1+x) - 5 = 0.$$

18. A house may be bought for \$2800 cash, or in annual installments of \$1000 each payable 1, 2, and 3 years from date. What is the annual rate of interest implied in this offer?

19. An open box is to be made from a rectangular piece of tin 18 inches long and 10 inches wide, by cutting out equal squares from the corners and turning up the sides. How large should these squares be in order that the box contain 168 cubic inches?

20. Find the cube root of 3.

HINT. — Find the approximate value of the real root of the equation $x^3 = 3$.

21. Find the cube root of -17.

22. Find the fifth root of 10.

23. Find the cube root of 115.

Find the points of intersection of the following curves:

24. $x^2 + y^2 = 10$, $y = x^2 + x + 1$.

25. $x^2 + y^2 = 9$, $y = x^2 - x$.

26. $4x^2 + y^2 = 16$, $x + 5y = y^2 + 2$.

CHAPTER XIV

DETERMINANTS

167. In Chapter V we used determinants of the second and third orders to advantage in the solution of systems of linear equations in two and three unknowns respectively. Analogous symbols, which are called **determinants of order *n*** can be used in the solution of systems of n linear equations in n unknowns, where n is any positive integer. For values of n greater than 3 there is a greater advantage in this use of determinants than there is in the simple cases in which $n = 1, 2$, or 3.

The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ was called a determinant of the second order and by definition $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$; likewise, by definition of a determinant of the third order,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

We wish to define a determinant of order n, where n is any positive integer, in a way that will be consistent with these definitions of determinants of order two and three respectively.

168. Inversions of numbers. — But we must first introduce the notion of an *inversion*. We use this term to describe the presence in an arrangement of positive integers of a greater integer before a smaller one.

Thus, in the arrangement 1, 2, 4, 5, 6, 3, the integers 4, 5, and 6 each appear before the smaller one 3, and there are in this arrangement therefore three inversions. There are no inversions in the arrangement 1, 2, 3, 4, 5, 6; and there are two in the arrangement 2, 1, 3, 5, 4, 6.

Theorem. — *If in an arrangement of positive integers any two of the integers be interchanged, the number of inversions is increased or diminished by an odd number.*

Consider first the effect of interchanging two adjacent integers a and b. Every integer that preceded both a and b in the original arrangement will precede them in the new arrangement, and every integer that followed them originally will follow them in the new arrangement. Moreover, the only relative positions that are changed are those of a and b, and the effect of this change of relative positions is to increase or decrease the number of inversions by 1 according as a is less than, or greater than, b. That is, the effect of the interchange of two adjacent integers is to increase or decrease the number of inversions by 1.

Suppose now that there are k integers between a and b. Then b can be brought to the original position of a by $k+1$ interchanges of adjacent integers, and after this has been done a can be brought to the original position of b by k interchanges of adjacent integers. These interchanges do not affect the relative positions of the integers other than a and b. The desired interchange of a and b has then been brought about by $2k+1$ interchanges of adjacent integers. In general, some of these interchanges, say x of them, have each caused an increase of 1 in the number of inversions and the remaining $2k+1-x$ have caused a decrease of 1 each. The net result has been an increase or a decrease of the number of inversions equal to the difference of these two numbers. But this difference, which is either $2k+1-2x$ or $2x-2k-1$, is an odd number for all possible values of x. The theorem is therefore proved.

169. Inversions of letters. — In an arrangement of letters of the alphabet the presence of a letter before one that precedes it in the alphabetical order is also called an inversion.

Consider now the symbols $a_1, a_2, a_3, \cdots$; $b_1, b_2, b_3, \cdots$; $c_1, c_2, c_3, \cdots$; and so on; and select a set of these in which neither any

letter nor any subscript occurs twice. We have then the following

Theorem. — *The number of inversions in the letters when there are no inversions in the subscripts and the number of inversions in the subscripts when there are no inversions in the letters are both even or both odd.*

Consider first the special case $a_4b_2c_1d_5e_3$. Here there are no inversions in the letters. If we interchange d_5 and e_3 in order to bring the letter with the greatest subscript to the last place, we get the arrangement $a_4b_2c_1e_3d_5$. In doing this we have changed the number of inversions in the letters and the number of inversions in the subscripts each by an odd number. If now we interchange a_4 and e_3 in order to bring the letter with the next to the greatest subscript to the next to the last place, we shall again change the number of inversions in the letters and the number of inversions in the subscripts each by an odd number. Finally, by the interchange of e_3 and c_1 we get the arrangement $c_1b_2e_3a_4d_5$, and this last step changed the number of inversions in the letters and the number of inversions in the subscripts each by an odd number. In the final arrangement there are no inversions in the subscripts. The number of inversions in the subscripts has been changed three times, by an odd number each time, and the number of inversions in the letters has been changed the same number of times, by an odd number each time. Hence the number of inversions in the subscripts in the original arrangement and the number of inversions in the letters in the final arrangement are both odd.

In general we can change the arrangement in which there are no inversions in the letters to the arrangement in which there are no inversions in the subscripts by a certain number of interchanges of letters. Each of these interchanges increases or diminishes the number of inversions in the subscripts by an odd number (§ 168), and also increases or diminishes the number of inversions in the letters by an odd number. Hence the number of inversions in the letters when there are no inver-

sions in the subscripts and the number of inversions in the subscripts when there are no inversions in the letters are both even or both odd according as the number of these successive interchanges is even or odd.

170. Definition of a determinant. — *In an array of n^2 numbers in n rows and n columns form all possible products of n factors each, taking as factors one number, and but one, from each row and column.*

Arrange the factors of each product in the order of the columns in which they occur and change the signs of those products in which the arrangement of the integers representing the respective rows presents an odd number of inversions.

The algebraic sum of these products with their signs thus modified is called a determinant of the nth order. It is represented by the array inclosed between two vertical lines.

Thus, $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$$

$$\begin{vmatrix} 2 & 4 & -1 \\ 3 & 6 & 2 \\ 4 & 7 & 5 \end{vmatrix}$$

$$= 2 \cdot 6 \cdot 5 + 3 \cdot 7 \cdot (-1) + 4 \cdot 4 \cdot 2 - 4 \cdot 6 \cdot (-1) - 2 \cdot 7 \cdot 2 - 3 \cdot 4 \cdot 5 = 7.$$

Here, for example, the sign of the product $3 \cdot 4 \cdot 5$ is changed because when its factors are arranged in the order of the columns in which they occur there is an inversion in the order of the rows in which they occur, this order being 2, 1, 3.

171. Definitions. — The products described in the definition with their proper signs are called the **terms** of the determinant.

When a determinant is written out in full as the algebraic sum of its terms, it is said to be **expanded.**

The numbers in the array from which the terms are formed are called the **elements** of the determinant.

172. Properties of determinants. — I. *There are $n!$ terms in the expansion of a determinant of order n.*

This is an immediate consequence of the fact that there are $n!$ permutations of the n rows taken n at a time (see § 104).

When n is greater than 3, $n!$ is so large that it is not practicable to find the value of the determinant by the direct process of expansion. And besides the number of the terms there is the difficulty of determining which of the products described in the definition should have their signs changed. These difficulties can be avoided by the application of some of the following properties of determinants.

II. *A determinant is not changed if the corresponding columns and rows are interchanged.*

Since each term of a determinant of order n is the product, with a possible change of sign, of n factors, no two of which occur in the same row or the same column, it is evident that the interchange of corresponding rows and columns will have no effect on the terms except perhaps to change the signs of some of them.

If the original form of the determinant is

$$\begin{vmatrix} a_1 & b_1 & c_1 \cdots \\ a_2 & b_2 & c_2 \cdots \\ \cdot & \cdot & \cdot \cdot \cdot \\ \cdot & \cdot & \cdot \cdot \cdot \\ a_n & b_n & c_n \cdots \end{vmatrix},$$

any of its terms is a product of n factors so arranged that there are no inversions in the letters and the sign of this product is changed or not according as the number of inversions in the subscripts is odd or even. The corresponding term of the new determinant is the product of the same factors so arranged that there are no inversions in the subscripts and the sign of the product is changed or not according as the number of inversions in the letters is odd or even. But by the theorem of § 169, the number of inversions in the subscripts is

here odd or even according as the number of inversions in the letters is odd or even. Hence the determinant is not changed by the interchange of corresponding rows and columns.

The student should verify directly that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

It follows from II that for every theorem concerning the rows of a determinant there is an analogous theorem concerning its columns.

III. *If all the elements of a row (or column) of a determinant are multiplied by the same number, the value of the determinant is multiplied by this number.*

This follows from the fact that one, and only one, element from this row (or column) is a factor of every term of the expansion of the determinant.

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = ka_1b_2c_3 + ka_2b_3c_1 + ka_3b_1c_2 - ka_3b_2c_1 - ka_1b_3c_2 - ka_2b_1c_3$$

$$= k(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3) = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

IV. *The sign of a determinant is changed if two of its rows (or columns) are interchanged.*

This is an immediate consequence of the definition of a determinant and the theorem of § 168.

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} = a_1b_3c_2 + a_3b_2c_1 + a_2b_1c_3 - a_2b_3c_1 - a_3b_1c_2 - a_1b_2c_3$$

$$= -(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3) = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

V. *The value of a determinant is zero if the corresponding elements of two of its rows (or columns) are the same.*

Let D represent the determinant.

Then by IV D is changed into $-D$ by the interchange of any two rows (or columns). On the other hand, if these two rows (or columns) are the two identical ones, D is not affected by their interchange. Hence

$$D = -D,$$
$$2D = 0,$$

and

$$D = 0.$$

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1 - a_3b_1c_1 - a_1b_3c_1 - a_1b_1c_3 = 0.$$

VI. *The value of the determinant is zero if the elements of any row (or column) are m times the corresponding elements of any other row (or column).*

We know from III that the value of such a determinant is m times the value of a determinant the corresponding elements of two of whose rows (or columns) are the same, and we know from V that the value of this latter determinant is zero.

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ma_2 & mb_2 & mc_2 \end{vmatrix} = ma_1b_2c_2 + ma_2b_2c_1 + ma_2b_1c_2 - ma_2b_2c_1 - ma_1b_2c_2 - ma_2b_1c_2 = 0.$$

VII. *If two determinants are the same except possibly for the elements of a certain row (or column), their sum is equal to a determinant in which the elements of this row (or column) are the sums of the corresponding elements of the two determinants and the elements of the other rows are the same as in the two determinants.*

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 \\ a'_2 & b_2 & c_2 \\ a'_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + a'_1 & b_1 & c_1 \\ a_2 + a'_2 & b_2 & c_2 \\ a_3 + a'_3 & b_3 & c_3 \end{vmatrix}.$$

That this is so is obvious when we consider that every term of the third determinant is the sum of two parts which are respectively the corresponding terms of the first two determinants.

VIII. *The value of a determinant is not changed if each element of any row (or column) multiplied by any number m be added to the corresponding elements of any other row (or column).*

By VII the resulting determinant is equal to the sum of two determinants of which one is the original determinant and the other has the value zero by VI.

For example,

$$\begin{vmatrix} a_1 + mc_1 & b_1 & c_1 \\ a_2 + mc_2 & b_2 & c_2 \\ a_3 + mc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mc_1 & b_1 & c_1 \\ mc_2 & b_2 & c_2 \\ mc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

173. Minors of a determinant. — If in a determinant of order n we omit the elements of any row and any column, the remaining elements with their relative positions unchanged form a determinant of order $n-1$ which is called the **minor** of the element in this row and this column.

Thus, if in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

we omit the elements of the second row and the third column the remaining elements form the determinant

$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix},$$

which is the minor of c_2.

If an element of a determinant is represented by a small letter with a subscript, we represent its minor by the corresponding capital letter with the same subscript.

174. The expansion of a determinant of the third order can be arranged as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1A_1 - a_2A_2 + a_3A_3.$$

That is, a determinant of the third order can be expressed in terms of determinants of the second order. In a similar way a determinant of order n can be expressed in terms of determinants of order $n - 1$. The following theorem states in detail how this can be done.

Theorem. — *Multiply the elements of any row (or column) of a determinant by their respective minors, and if the sum of the number of the row and the number of the column in which the element occurs is odd, change the sign of the resulting product. The algebraic sum of these products with their signs thus changed is equal to the determinant.*

Represent the determinant by D, the element in the upper left hand corner by a_1, and any other element by s_i.

1. If the elements in the terms of D and A_1 are arranged as described in § 170, the coefficient of a_1 in any term of D is a term of A_1, since the row and column in which a_1 occurs precede all the other rows and columns. This conclusion is based upon the fact that in any arrangement of a series of integers beginning with 1 the number of inversions is the same as it is in this arrangement with the 1 omitted. Thus 135462 and 35462 contain the same number of inversions.

Conversely, any term of A_1 is the coefficient of a_1 in some term of D.

For example, in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ the coefficient of a_1 is $b_2c_3 - b_3c_2$, which is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, or A_1.

2. Consider the element s_i in the ith row and jth column of D. It can be brought to the first row by $i-1$ interchanges of adjacent rows, and then to the first column by $j-1$ interchanges of adjacent columns, without affecting the relative positions of the elements of D not contained in the original ith row or the original jth column. The effect of these interchanges is to change the sign of the determinant $(i-1+j-1)$ times; that is, $(i+j-2)$ times. This number is odd or even according as $i+j$ is odd or even. If then D' denotes the determinant resulting from these changes,

$$D' = (-1)^{i+j} D.$$

If we multiply both members of this equation by $(-1)^{i+j}$, we get

$$(-1)^{i+j} D' = (-1)^{2i+2j} D = D,$$

since $(-1)^{2i+2j} = 1$.

Since now the relative positions of the elements not in the ith row or the jth column of D are not affected by these changes, the minor of s_i in D' is the same as its minor in D. And from (1) we see that the coefficient of s_i in D' is S_i. Hence the coefficient of s_i in D is $(-1)^{i+j} S_i$.

For example, in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ consider the element b_3. Here $i = 3$ and $j = 2$.

Now $$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_3 & b_3 & c_3 & d_3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = - \begin{vmatrix} b_3 & a_3 & c_3 & d_3 \\ b_1 & a_1 & c_1 & d_1 \\ b_2 & a_2 & c_2 & d_2 \\ b_4 & a_4 & c_4 & d_4 \end{vmatrix}.$$

From (1) we know that the coefficient of b_3 in this last determinant is $\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_4 & c_4 & d_4 \end{vmatrix}$, and therefore the coefficient of b_3 in the original determinant is $-\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$.

We shall get all the terms in the expansion of D by taking the sum of all the terms containing each of the elements of a given row or column, and no term will appear more than once, since no term in the expansion of D can contain two elements from the same row or column.

This proves the theorem.

In order to see clearly the meaning of this theorem consider its application to a determinant of the fourth order:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

$$= a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

175. By means of successive applications of the theorem of the preceding article we can make the expansion of a determinant depend finally upon the expansion of a series of determinants of the third order. This process, simplified by means of the properties given in § 172 in a way we shall explain by means of examples, is the one most frequently used for the expansion of a determinant.

Example 1. — Expand the determinant

$$\begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 3 & 1 & 4 \\ 7 & 7 & 3 & 2 \\ 6 & 6 & 5 & -1 \end{vmatrix}.$$

If we subtract the elements of the second column from the corresponding elements of the first column, the resulting determinant is, by § 172, VIII, where $m = -1$, equal to D. Hence

$$D = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 7 & 3 & 2 \\ 0 & 6 & 5 & -1 \end{vmatrix}.$$

Hence, by § 174,

$$D = \begin{vmatrix} 3 & 1 & 4 \\ 7 & 3 & 2 \\ 6 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ -2 & 3 & -10 \\ -9 & 5 & -21 \end{vmatrix} = -42 + 90 = 48.$$

The last determinant here is obtained from the preceding one by an application of § 172, VIII.

The advantage of the preliminary transformation is due to the fact that it puts D into a form in which all but one of the elements of a certain column are zero. In the expansion of a determinant in such a form by § 174 all the terms except the first one are equal to zero.

Example 2. — Expand the determinant

$$D = \begin{vmatrix} 9 & 13 & 17 & -4 \\ 18 & 27 & 35 & -8 \\ 30 & 44 & 21 & 10 \\ 12 & 15 & 0 & 2 \end{vmatrix}.$$

$$D = \begin{vmatrix} 9 & 13 & 17 & -4 \\ 0 & 1 & 1 & 0 \\ 30 & 44 & 21 & 10 \\ 12 & 15 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 9 & 13 & 4 & -4 \\ 0 & 1 & 0 & 0 \\ 30 & 44 & -23 & 10 \\ 12 & 15 & -15 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 9 & 4 & -4 \\ 30 & -23 & 10 \\ 12 & -15 & 2 \end{vmatrix} = 3 \begin{vmatrix} 3 & 4 & -4 \\ 10 & -23 & 10 \\ 4 & -15 & 2 \end{vmatrix} = 3 \begin{vmatrix} 3 & 4 & 0 \\ 10 & -23 & -13 \\ 4 & -15 & -13 \end{vmatrix}$$

$$= 3(897 - 208 - 585 + 520) = 1872.$$

Here we multiplied the elements of the first row of the original determinant by 2 and subtracted the products from the corresponding elements

of the second row. Then we subtracted the elements of the second column of the resulting determinant from the corresponding elements of the third column.

If the determinant has an element equal to ± 1, we can easily, by applying § 172, VIII, make all the other elements in the same row, or the same column, zero. Then by the theorem of § 174 the determinant is equal to a determinant of lower order.

If the determinant has no element equal to ± 1, a suitable application of § 172, VIII, will transform it into one with at least one element equal to ± 1.

When we get to determinants of the third order, it is best to expand them in full unless there is an obvious way to shorten the process.

The student should always be on the lookout to see if the elements of any row (or column) are equal to m times the corresponding elements of any other row (or column). In this case the determinant equals zero (§ 172, VI).

If the elements of any row (or column) are integers with a common factor, § 172, III, can be applied to advantage.

176. Theorem. — *If the elements of a determinant D are polynomials in x, and if $D = 0$ when $x = a$, then $x - a$ is a factor of the expansion of D.*

This is an immediate consequence of the Factor Theorem (§ 142, Cor.), since the expansion of D is a polynomial in x.

For example, if
$$D = \begin{vmatrix} x^2 & x & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix},$$

it is evident that $D = 0$ when $x = 1$ and when $x = 2$ (§ 172, V). Hence D is divisible by $x - 1$ and by $x - 2$. Moreover it is clear from § 174 that D is of the second degree in x, and that the coefficient of x^2 is

$$\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1.$$

Hence
$$\begin{vmatrix} x^2 & x & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (x-1)(x-2).$$

EXERCISES

1. How many inversions are there in the arrangement 1, 7, 5, 8, 6, 2, 4, 3? In the arrangement 7, 5, 8, 6, 2, 4, 3?

2. How many inversions are there in the subscripts of $a_2\ b_5\ c_6\ d_1\ e_7\ f_4\ g_3$? How many inversions in the letters of $d_1\ a_2\ g_3\ f_4\ b_5\ c_6\ e_7$? Compare § 169.

3. Determine by inspection whether the following two determinants are equal or not:

$$\begin{vmatrix} 3 & -4 & 0 & 1 \\ 6 & 2 & 7 & 9 \\ 1 & 3 & 8 & 4 \\ -6 & 0 & 7 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 3 & 6 & 1 & -6 \\ -4 & 2 & 3 & 0 \\ 0 & 7 & 8 & 7 \\ 1 & 9 & 4 & 3 \end{vmatrix}.$$

4. Show without expanding the determinants that

$$\begin{vmatrix} 3 & 0 & 14 & 2 \\ 7 & 5 & 4 & 9 \\ 8 & -10 & 1 & 6 \\ 5 & 8 & 3 & 12 \end{vmatrix} + \begin{vmatrix} 3 & 0 & 2 & 14 \\ 7 & 5 & 9 & 4 \\ 8 & -10 & 6 & 1 \\ 5 & 8 & 12 & 3 \end{vmatrix} = 0.$$

5. Express the following sum as a single determinant of the fourth order:

$$\begin{vmatrix} 15 & 2 & 6 & 1 \\ -5 & 0 & 4 & 8 \\ 2 & 7 & 3 & 10 \\ 8 & 8 & 2 & 2 \end{vmatrix} + \begin{vmatrix} -10 & 2 & 6 & 1 \\ 5 & 0 & 4 & 8 \\ 7 & 7 & 3 & 10 \\ 2 & 8 & 2 & 2 \end{vmatrix}.$$

Find the values of the following determinants:

6. $\begin{vmatrix} 3 & 15 & 16 & 29 \\ 6 & 7 & 33 & 14 \\ 6 & 30 & 32 & 58 \\ 4 & 2 & 5 & 6 \end{vmatrix}$.

7. $\begin{vmatrix} 3 & 4 & -2 \\ 4 & -3 & 8 \\ 2 & 8 & -3 \end{vmatrix}$.

8. $\begin{vmatrix} 3 & 4 & 10 & 2 \\ 0 & -6 & 8 & 7 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \end{vmatrix}$.

10. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 4 & 5 & 2 \\ 3 & 2 & 4 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix}$.

9. $\begin{vmatrix} 4 & 7 & 6 & 5 \\ 2 & 1 & 3 & 7 \\ 2 & 5 & 3 & 4 \\ 14 & 4 & 6 & 5 \end{vmatrix}$.

11. $\begin{vmatrix} \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & \frac{3}{2} \\ 4 & 0 & \frac{3}{2} & \frac{1}{2} \\ 5 & 2 & 6 & 1 \\ 0 & 1 & 4 & 2 \end{vmatrix}$.

Factor the following determinants:

12. $\begin{vmatrix} x^3 & x^2 & x & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ \frac{8}{27} & \frac{4}{9} & \frac{2}{3} & 1 \end{vmatrix}$.

13. $\begin{vmatrix} 3+x & 4-x & x+1 \\ 5 & 2 & 3 \\ 0 & 6 & 1 \end{vmatrix}$.

14. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

is divisible by $x-y$, $y-z$, and $z-x$.

15. Show that $\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 4 & 1 \end{vmatrix} = 0$

when $x=2$, $y=3$; and also when $x=-1$, $y=4$.

16. Show that $\begin{vmatrix} x & y & 1 \\ -1 & -2 & 1 \\ 3 & 5 & 1 \end{vmatrix} = 0$

is the equation of the straight line through the points $(-1, -2)$ and $(3, 5)$.

17. Find three sets of values of x, y, and z for which

$$\begin{vmatrix} x & y & z & 1 \\ 5 & 3 & -2 & 1 \\ 4 & 1 & 1 & 1 \\ 0 & 0 & 6 & 1 \end{vmatrix} = 0.$$

177. Solutions of systems of linear equations by means of determinants.— It was stated at the beginning of this chapter that determinants of order n could be used in the solution of systems of n linear equations in n unknowns. We are now in a position to see in detail how this is done.

We shall consider the solution of a system of four linear equations in four unknowns. The method used is applicable to the general case.

Let the given equations be

$$a_1x + b_1y + c_1z + d_1w = k_1, \quad (1)$$
$$a_2x + b_2y + c_2z + d_2w = k_2, \quad (2)$$
$$a_3x + b_3y + c_3z + d_3w = k_3, \quad (3)$$
$$a_4x + b_4y + c_4z + d_4w = k_4. \quad (4)$$

The determinant formed by the coefficients of the unknowns, that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

is called the determinant of the system of equations. We shall represent it by D, and shall assume that the equations are such that $D \neq 0$. If there is a set of values of x, y, z, and w that satisfy these equations, these values can be found in the following way:

Multiply each member of equations (1), (2), (3), and (4) by A_1, $-A_2$, A_3, and $-A_4$ respectively. This gives

$$a_1A_1x + b_1A_1y + c_1A_1z + d_1A_1w = k_1A_1, \quad (5)$$
$$-a_2A_2x - b_2A_2y - c_2A_2z - d_2A_2w = -k_2A_2, \quad (6)$$
$$a_3A_3x + b_3A_3y + c_3A_3z + d_3A_3w = k_3A_3, \quad (7)$$
$$-a_4A_4x - b_4A_4y - c_4A_4z - d_4A_4w = -k_4A_4. \quad (8)$$

Putting the sum of the left members of equations (5), (6), (7), and (8) equal to the sum of their right members, we get

$$\begin{aligned} &x(a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4) + y(b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4) \\ &\quad + z(c_1A_1 - c_2A_2 + c_3A_3 - c_4A_4) \\ &\quad + w(d_1A_1 - d_2A_2 + d_3A_3 - d_4A_4) \\ &\quad = k_1A_1 - k_2A_2 + k_3A_3 - k_4A_4. \end{aligned} \tag{9}$$

Now the coefficient of x in equation (9) is, by § 174, equal to D, and the coefficients of y, z, and w are

$$\begin{vmatrix} b_1 & b_1 & c_1 & d_1 \\ b_2 & b_2 & c_2 & d_2 \\ b_3 & b_3 & c_3 & d_3 \\ b_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} c_1 & b_1 & c_1 & d_1 \\ c_2 & b_2 & c_2 & d_2 \\ c_3 & b_3 & c_3 & d_3 \\ c_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} d_1 & b_1 & c_1 & d_1 \\ d_2 & b_2 & c_2 & d_2 \\ d_3 & b_3 & c_3 & d_3 \\ d_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

respectively. But these last three determinants are, by § 172, V, equal to zero.

Hence this equation can be written in the form

$$Dx = k_1A_1 - k_2A_2 + k_3A_3 - k_4A_4. \tag{10}$$

In a similar way we get

$$Dy = -k_1B_1 + k_2B_2 - k_3B_3 + k_4B_4, \tag{11}$$

$$Dz = k_1C_1 - k_2C_2 + k_3C_3 - k_4C_4, \tag{12}$$

$$Dw = -k_1D_1 + k_2D_2 - k_3D_3 + k_4D_4. \tag{13}$$

Now the right members of these four equations are equal to

$$\begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & k_1 & d_1 \\ a_2 & b_2 & k_2 & d_2 \\ a_3 & b_3 & k_3 & d_3 \\ a_4 & b_4 & k_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}$$

respectively.

Therefore, if there is a set of values of x, y, z, and w that satisfy equations (1), (2), (3), and (4), these values must be given by the formulæ

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}}{D}, \qquad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix}}{D},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 & d_1 \\ a_2 & b_2 & k_2 & d_2 \\ a_3 & b_3 & k_3 & d_3 \\ a_4 & b_4 & k_4 & d_4 \end{vmatrix}}{D}, \qquad w = \frac{\begin{vmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}}{D}. \tag{14}$$

It is now necessary to show that these values do satisfy equations (1), (2), (3), and (4). If we substitute these values for x, y, z, and w respectively in the left member of (1), we get

$$\frac{a_1\begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix} + b_1\begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_1 & b_1 & k_1 & d_1 \\ a_2 & b_2 & k_2 & d_2 \\ a_3 & b_3 & k_3 & d_3 \\ a_4 & b_4 & k_4 & d_4 \end{vmatrix} + d_1\begin{vmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}}{D}.$$

And this, when arranged with reference to k_1, k_2, k_3, and k_4 becomes

$$\frac{\begin{aligned} &k_1(a_1A_1 - b_1B_1 + c_1C_1 - d_1D_1) + k_2(-a_1A_2 + b_1B_2 - c_1C_2 + d_1D_2) \\ &\quad + k_3(a_1A_3 - b_1B_3 + c_1C_3 - d_1D_3) \\ &\quad + k_4(-a_1A_4 + b_1B_4 - c_1C_4 + d_1D_4) \end{aligned}}{D}.$$

In the numerator of this fraction the coefficient of k_1 is D and the coefficients of k_2, k_3, and k_4 are

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_1 & b_1 & c_1 & d_1 \end{vmatrix}$$

respectively. Each of these last three determinants is equal to zero. Hence the fraction is equal to k_1. That is, these values of x, y, z, and w satisfy equation (1). In a similar way

it can be shown that these values also satisfy equations (2), (3), and (4).

The student should observe that these formulæ for the solution of equations (1), (2), (3), and (4) are closely analogous to the formulæ for the solution of systems of two or three linear equations in two or three unknowns respectively. They apply with slight and obvious modifications to the solutions of systems of n linear equations in n unknowns.

178. Inconsistent equations. — The argument by means of which these solutions were obtained breaks down in case $D=0$. In this case the left members of equations (10), (11), (12), and (13), § 177, are all zero. Hence, unless the numerators of formula (14) are all zero, we have a contradiction, and the original equations, (1), (2), (3), and (4), are *inconsistent*. This argument can be extended to a system of n linear equations in n unknowns. We conclude therefore that *if the determinant D of a system of n linear equations in n unknowns is zero, the equations are inconsistent unless all the determinants in the formulæ analogous to* (14) *are zero.*

Example. — Show that the equations

$$\begin{aligned} 7x + 6y + 2z - w &= -2, \\ -2x + 4y + 5z + 2w &= 3, \\ 3x - 5y - 6z + 4w &= 6, \\ 8x + 5y + z + 5w &= 4, \end{aligned}$$

are inconsistent.

The equations may be inconsistent when D and all the determinants in the formulæ analogous to (14) are zero. This question is, however, too complicated to be discussed here.

179. Homogeneous equations. — Equations (1), (2), (3), and (4) are said to be **homogeneous** when their right members are all zero. In this case, if $D \neq 0$, the only values of the unknowns that satisfy the equations are $x=0$, $y=0$, $z=0$, and $w=0$. If $D=0$, there are other solutions. An analogous thing holds for a system of n linear homogeneous equations in n unknowns.

EXERCISES

Solve the following systems of equations by means of formulæ (14) or similar formulæ:

1. $2x - y + z = 5,$
   $y - 2z - 5w = -8,$
   $x + y + z + w = 2,$
   $x - 2y + 5w = 6.$

2. $5x - 2y + z + 3w - 7 = 0,$
   $2x + 4y + 3z + 2w = 8,$
   $x - y - z + w = 4,$
   $3x + y + 5z + 4w = 9.$

3. $x + y = a,$
   $y + z = b,$
   $z + w = c,$
   $2w + x = d.$

4. $2x + 3y - z + 3 = 0,$
   $2x - 3y + 3z = 2,$
   $-x + 2y + 5z = 5.$

5. $3A + 2B - C + D = 2,$
   $A + 3B + C - D = 5,$
   $A + B = 4,$
   $C + D = 6.$

6. $-4x + 5y + 4z - 3w = 0,$
   $7x + y + z + w = 34,$
   $4x - 10y + 4z - w = 6,$
   $x - 6y - 8z + 5w = -19.$

7. $3x + y + w = 20,$
   $3z + 6x + w = 40,$
   $3z + x + 4y = 30,$
   $5z + 8y + 3w = 50.$

8. $A + 2B + 3C + 4D = 1,$
   $4A + B + 2C + 3D + 3 = 0,$
   $3A + 4B + C + 2D + 6 = 0,$
   $2A + 3B + 4C + D + 2 = 0.$

9. $A + 2B + 3C - D - 2E = 5,$
   $2A + B + 3C - 2D + E = 4,$
   $B + 2C - 2D + E = 2,$
   $-B - C + 2D + E = 0,$
   $2B + 3C + D - E = 6.$

10. $x + y + z + w = a,$
    $x - y + z - w = b,$
    $2x + 3y - 2w = l,$
    $y + z = m.$

11. $3x + 2y - 5z = 5,$
    $4x - 6y + 2z = -10,$
    $2x + 10y - 12z = 8.$

12. $A - 5B - 6C = 2,$
    $-3A - 4B + 10C = 7,$
    $10A + 2B - 36C = -1.$

CHAPTER XV

INEQUALITIES

180. Definition of inequality. — We have considered in some detail how to identify numbers that are described by means of equations. In certain important problems, however, the information we are given concerning the numbers involved is more indefinite than that contained in equations. It merely tells us that one thing is greater or less than another. Such a statement is called an **inequality.**

Inequalities are used only in connection with real numbers, and the real number a is greater than the real number b if $a-b$ is positive, and a is less than b if $a-b$ is negative. (§ 3.)

The expression $a \geqq b$ means that a is greater than, or equal to, b, and $a \leqq b$ means that a is less than, or equal to, b.

Two inequalities like $a > b$ and $c > d$, in each of which the greater number appears on the same side, are said to be *alike in sense.* On the other hand, $a > b$ and $c < d$ are said be *opposite in sense.*

181. Conditional and unconditional inequalities. — Some inequalities are satisfied by all real numbers and are therefore called **unconditional** inequalities. They are analogous to identical equations.

Thus $x^2 + 1 > 0$ is an unconditional inequality. Such a description gives us no information whatever about the numbers involved.

Many inequalities on the other hand are satisfied only by certain numbers. These are called **conditional** inequalities.

Thus $x + 1 > 0$ is a conditional inequality, since not every real number when added to 1 gives a sum greater than 0.

182. Properties of inequalities. — In order to identify as closely as possible the numbers described by means of an inequality we proceed very much as in the solution of an equation. The justification of such a procedure is found in the following theorems:

I. *The sense of an inequality is not changed if both sides are increased or decreased by the same number.*

For if $a > b$,

then $a - b = k$, where k is a positive number,

and $a + n - b - n = k$, where n is any real number, positive or negative;

or $(a + n) - (b + n) = k$.

Hence $a + n > b + n$.

This demonstration applies, of course, to inequalities of the opposite sense since the statement that $a > b$ is equivalent to the statement that $b < a$.

It follows that terms can be transposed from one side of an inequality to the other by changing their signs, as in the case of equations.

Thus, the conditional inequality $x + 1 > 0$ that was referred to in § 181 is equivalent to the inequality $x > -1$.

II. *The sense of an inequality is not changed if both sides are multiplied or divided by the same positive number.*

For if $a > b$, and m is any positive number,

then $a - b = k$, where k is a positive number,

and $am - bm = km$.

But since k and m are both positive, km is positive, and therefore $am > bm$.

III. *The sense of an inequality is reversed if both sides are multiplied or divided by the same negative number.*

The proof of this is left to the student.

It follows from Theorem III that the sense of an inequality is reversed if the signs of all its terms are changed.

Thus, if $a > b$,

then $-a < -b$.

183. Conditional inequalities. — For our purpose the most important conditional inequalities are those involving one unknown to the first or second degree. These are called **linear** and **quadratic** inequalities respectively.

By Theorem I of the preceding article, every inequality involving one unknown can be put into the form $f(x) > 0$.

If $f(x)$ is linear in x, the inequality is of the form

$$ax + b > 0.$$

Hence $$x > -\frac{b}{a},$$

or $$x < -\frac{b}{a},$$

according as a is positive or negative.

If $f(x)$ is a quadratic in x, we have

$$ax^2 + bx + c > 0.$$

Now by § 148, $ax^2 + bx + c = a(x - r_1)(x - r_2)$,

where r_1 and r_2 are the roots of the equation

$$ax^2 + bx + c = 0.$$

If r_1 and r_2 are imaginary, $ax^2 + bx + c$ has the same sign for all values of x, since if it had opposite signs for $x = l$ and $x = m$, the equation would have a real root between l and m (§ 146). This sign is the same as the sign of c inasmuch as $ax^2 + bx + c = c$ when $x = 0$.

Since the condition for imaginary roots is (§ 71) $b^2 - 4ac < 0$, we see that a and c must have the same sign when the roots are imaginary. Hence in this case $ax^2 + bx + c$ has the same sign as a for all values of x.

If $r_1 = r_2$, $(b^2 - 4ac = 0)$, we have

$$ax^2 + bx + c = a(x - r_1)^2.$$

Hence $ax^2 + bx + c$ has the same sign as a for all values of x, except $x = r_1$. For this value

$$ax^2 + bx + c = 0.$$

If r_1 and r_2 are real $(b^2 - 4ac > 0)$ and $r_1 < r_2$ we have

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Then when a is positive $ax^2 + bx + c$ is negative for all values of x between r_1 and r_2 and positive for all other values except r_1 and r_2. When a is negative, $ax^2 + bx + c$ is positive for all values of x between r_1 and r_2, and negative for all other values except r_1 and r_2.

The following table exhibits these results in convenient form:

a	$b^2 - 4ac$	$ax^2 + bx + c$
+	−	+
+	0	+, except for $x = r_1$.
+	+	− for values of x between r_1 and r_2; + for all other values, except r_1 and r_2.
−	−	−
−	0	−, except for $x = r_1$.
−	+	+ for values of x between r_1 and r_2; − for all other values, except r_1 and r_2.

Example. — For what values of x is $x^2 - 10 > 5 + x - x^2$?

By Theorem I this inequality is equivalent to

$$2x^2 - x - 15 > 0.$$

Here $r_1 = -\frac{5}{2}$, $r_2 = 3$, and $a = 2$. Hence the given inequality is satisfied by all values of x less than $-\frac{5}{2}$ and by all values greater than 3.

We can treat quadratic inequalities by completing the square, as is done in solving quadratic equations, if we observe proper precautions. If we treat the preceding example in this way, we get

$$x^2 - \tfrac{1}{2}x + \tfrac{1}{16} > \tfrac{121}{16}.$$

But we cannot conclude from this that

$$x - \tfrac{1}{4} > \pm \tfrac{11}{4}.$$

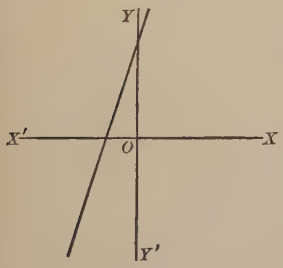
All we can conclude is that either

$$x - \tfrac{1}{4} > \tfrac{11}{4} \text{ or } x - \tfrac{1}{4} < -\tfrac{11}{4},$$

and these two inequalities lead to the results already obtained.

184. Graphical interpretation of linear and quadratic inequalities. — The values of x for which $ax + b > 0$ are the abscissæ of the points on the locus of the equation

$$y = ax + b$$



that are above the x-axis, and the values of x for which $ax + b < 0$ are the abscissæ of the points on this locus that are below the x-axis. The value of x for which $y = ax + b$ is the abscissa of the point of intersection of this locus and the x-axis.

Thus, a glance at the locus of the equation

$$y = 3x + 5$$

tells us for what values of x the expression

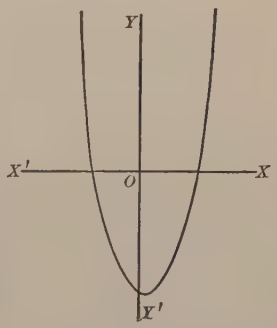
$$3x + 5 > 0, = 0, \text{ or } < 0.$$

In a similar way, an inspection of the locus of the equation

$$y = ax^2 + bx + c$$

tells us for what values of x the expression

$$ax^2 + bx + c > 0, = 0, \text{ or } < 0.$$



Suppose we wished to solve graphically the inequality considered in the latter part of the preceding article. We draw the locus of the equation $y = 2x^2 - x - 15$ and note the abscissæ of the points on the locus that are above the x-axis.

EXERCISES

1. Does the proof given in Theorem II of § 182 cover the case in which both members of the inequality are divided by the same positive number?

2. Prove that $x^2 + y^2 > 2xy$,

whenever x and y are unequal real numbers.

HINT. — This inequality is by Theorem I equivalent to

$$x^2 - 2xy + y^2 > 0.$$

What if x and y are equal?

3. Show that $x + \frac{1}{x} > 2$

for every positive value of x, except 1.

What is the relation between the two members of this inequality when $x = 1$?

4. Prove that if x, y, and z are unequal positive numbers

$$x^2 + y^2 + z^2 > xy + yz + zx.$$

5. Which is the greater, the arithmetic mean, or the positive geometric mean, of two unequal positive numbers?

Find for what values of x the following inequalities hold and interpret your results graphically:

6. $2x - 5 < x + 2$.

7. $\begin{vmatrix} x & 2 & -1 \\ 3x & 4 & 0 \\ 5 & 2 & 1 \end{vmatrix} < \begin{vmatrix} x & 3x \\ 2 & 5 \end{vmatrix}$.

8. $7x + 2x > 3x^2 + 5x + 1 + 4x^2 - 3x$.

9. $5x + 6 < 0$.

10. $x^2 + 2x + 5 \begin{cases} > 0. \\ = 0. \\ < 0. \end{cases}$

11. $-4x^2 + 20x - 25 < 0$.

12. $x^2 > 4$.

13. $(2x - 3)^2 < 9$.

14. $x(x - 2) \begin{cases} < 0. \\ = 0. \\ > 0. \end{cases}$

15. $\begin{vmatrix} 3 & 2 & -1 \\ x & 2x & x+1 \\ -5x & x-2 & 3x \end{vmatrix} > \begin{vmatrix} x & 2x & 0 \\ 4 & 7 & 1 \\ 0 & -2 & -6 \end{vmatrix}$.

16. $(2x+5)(x-2)\begin{cases} >0. \\ =0. \\ <0. \end{cases}$

17. $(x-2)^2\begin{cases} >1. \\ =1. \\ <1. \end{cases}$

18. $25x^2+18x+90<0$.

19. $(x-1)(x-2)(x-3)>0$.

20. $x(x+1)^2<0$.

21. $10-x>2x^2$.

22. $\dfrac{x+3}{x+2}>0$.

23. $\dfrac{2x-1}{3x-4}<0$.

24. $\dfrac{3}{x-5}\begin{cases} >0. \\ =0. \\ <0. \end{cases}$

25. For what values of m will the line

$$y=mx+5$$

have no points in common with the circle

$$x^2+y^2=9\,?$$

26. For what values of b will the line

$$y=3x+b$$

have no points in common with the parabola

$$y^2=8x\,?$$

27. For what values of b will the loci of the equations

$$y=3x+b$$

and

$$x^2+y^2-2x+4y+1=0$$

have one or more points in common?

CHAPTER XVI

PARTIAL FRACTIONS

185. It is explained in elementary algebra how to express the sum of several rational fractions as a single fraction whose denominator is the lowest common multiple of the denominators of the given fractions.

In certain problems it becomes desirable to perform the inverse operation on rational fractions of a particular form — namely, to express a fraction whose numerator and denominator are polynomials in x as the sum of two or more fractions the lowest common multiple of whose denominators is the denominator of the given fraction. These fractions whose sum equals the given fraction are called **partial** fractions.

We shall consider in this chapter a systematic means of decomposing rational fractions of the kind described into partial fractions.

186. If the fraction is $\dfrac{f(x)}{g(x)}$ and the degree of $f(x)$ is equal to, or greater than, the degree of $g(x)$, we can perform the division indicated by the fraction. If the resulting quotient is the polynomial $Q(x)$ and the remainder is $R(x)$, we have (§ 15)

$$\frac{f(x)}{g(x)} \equiv Q(x) + \frac{R(x)}{g(x)}.$$

Since the remainder is of lower degree than the denominator, we see that any rational fraction in x whose numerator is of as high a degree as its denominator is equal to the sum of a polynomial and a fraction whose numerator is of lower degree than its denominator. In discussing the decomposition of a rational fraction we shall therefore assume that its numerator is of lower degree than its denominator.

187. Theorem. — *Any such fraction* $\frac{f(x)}{g(x)}$ *can be decomposed into the sum of fractions of the type* $\frac{A}{(x-a)^p}$, *where* A *is a constant and* $(x-a)^p$ *is a factor of* $g(x)$.

The proof of this theorem will be omitted. We shall here take it for granted and consider the problem of determining these partial fractions for a given fraction $\frac{f(x)}{g(x)}$.

188. Case I. — *When the linear factors of* $g(x)$ *are distinct.*

Example. — Decompose $\frac{2x+3}{x^2-5x+6}$ into partial fractions.

The numerator is of lower degree than the denominator, and the linear factors of the denominator are $x-2$ and $x-3$. We know therefore from the theorem of § 187, that

$$\frac{2x+3}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}, \tag{1}$$

where A and B are certain constants.

The problem is to find the values of these constants.

Clearing (1) of fractions, we get

$$2x+3 = A(x-3) + B(x-2). \tag{2}$$

Now the two members (1) must be equal for all values of x except 2 and 3 (for these exceptional values of x neither member of (1) has any value), and therefore the two members of (2) which are polynomials of the first degree are equal for all these values. Hence by § 149, Cor., the coefficients of the like powers of x in the two members of (2) must be equal. That is

$$A + B = 2,$$

and

$$-3A - 2B = 3.$$

Solving these two equations for A and B, we get

$$A = -7 \text{ and } B = 9.$$

Hence

$$\frac{2x+3}{x^2-5x+6} = \frac{-7}{x-2} + \frac{9}{x-3}.$$

The student should verify that this is correct.

If, in general, $g(x)$ is the product of n distinct linear factors,

or $$g(x) \equiv (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n),$$

then, by § 187,

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}. \tag{1}$$

Clearing of fractions, we get

$$\begin{aligned} f(x) = {} & A_1(a_2x + b_2)(a_3x + b_3) \cdots (a_nx + b_n) \\ & + A_2(a_1x + b_1)(a_3x + b_3) \cdots (a_nx + b_n) \\ & + \cdots + A_n(a_1x + b_1)(a_2x + b_2) \cdots (a_{n-1}x + b_{n-1}). \end{aligned} \tag{2}$$

Now the two members of (1) must be equal for all values of x except $-\frac{b_1}{a_1}, -\frac{b_2}{a_2}, \cdots, -\frac{b_n}{a_n}$; and therefore (2) must be true for all these values. Moreover the left member of (2) is of degree not greater than $n-1$ and the right member is of degree $n-1$. Then by § 149, Cor., the coefficients of like powers of x in these two members must be equal. Hence the n unknowns, $A_1, A_2, \cdots, A_n$ must satisfy the n linear equations obtained by equating the coefficients of these like powers. By virtue of the theorem of § 187 we know that these equations cannot be inconsistent, and that therefore they can be solved.

When we have solved these equations and substituted the resulting values of $A_1, A_2, \cdots, A_n$ in (1) we have the desired decomposition of $\frac{f(x)}{g(x)}$ into partial fractions.

EXERCISES

Decompose the following fractions into partial fractions:

1. $\frac{2x}{x^2-1}$.

2. $\frac{4x^2-2x+3}{(x-1)(5x-2)(2x+3)}$.

3. $\frac{2x-1}{12x^3+20x^2+3x}$.

4. $\frac{2x^3-5x^2+x+7}{x^4-5x^2+4}$.

5. $\dfrac{x^2+x+1}{x^2+6x+8}$. (See § 186.)

6. $\dfrac{3x^2}{(x^2+7x+6)(x-2)}$.

7. $\dfrac{5}{(2x+1)(x+2)(x+3)}$.

8. $\dfrac{2x+5}{(x^2-9)(7x+2)}$.

9. $\dfrac{2x^3+x^2-x+1}{(x-1)(x^2+3x+2)}$.

10. $\dfrac{3x-2}{(x^2+x-2)(x+3)}$.

11. $\dfrac{2x}{x^2-x-2}$.

12. $\dfrac{4}{(x+1)(x^2-6x+5)}$.

13. $\dfrac{x^2+13x+36}{x^4-13x^2+36}$.

14. $\dfrac{3x-4}{6x^2-x-2}$.

15. $\dfrac{11x-2}{3x^2+11x-20}$.

16. $\dfrac{x^2+4x+5}{x^3+6x^2+11x+6}$.

17. $\dfrac{x+1}{(x-a)(x-b)}$.

18. $\dfrac{x^2+2x+3}{(2x-5)(5x-2)}$.

19. $\dfrac{(x+3)(x-4)}{(x-1)(x+2)(x-5)}$.

20. $\dfrac{3x+5}{(4x+7)(2x+3)}$.

189. Case II. — *When the linear factors of $g(x)$ are not all distinct.*

Example. — Decompose $\dfrac{3x^2+2x-4}{(x^2+6x+9)(x-1)}$ into partial fractions.

The numerator is of lower degree than the denominator, and the factors of the denominator are $(x+3)^2$ and $x-1$. We know therefore from the theorem of § 187 that

$$\frac{3x^2+2x-4}{(x^2+6x+9)(x-1)} = \frac{A}{x-1} + \frac{B}{(x+3)^2} + \frac{C}{x+3}, \qquad (1)$$

since the three fractions in the right member represent all the types of partial fractions mentioned in the theorem.

Clearing of fractions, we get

$$3x^2+2x-4 = A(x+3)^2 + B(x-1) + C(x-1)(x+3). \qquad (2)$$

By an argument similar to the one used in Case I we see that the coefficients of like powers of x in the two members of (2) must be equal.

Hence
$$3 = A + C,$$
$$2 = 6A + B + 2C,$$
$$-4 = 9A - B - 3C;$$

and
$$A = \tfrac{1}{16},\ B = -\tfrac{17}{4},\ C = \tfrac{47}{16}.$$

This gives
$$\frac{3x^2 + 2x - 4}{(x^2 + 6x + 9)(x - 1)} = \frac{\frac{1}{16}}{x - 1} - \frac{\frac{17}{4}}{(x + 3)^2} + \frac{\frac{47}{16}}{x + 3}$$

as the desired decomposition.

If, in general,
$$g(x) = (a_1x + b_1)^{n_1}(a_2x + b_2)^{n_2} \cdots (a_kx + b_k)^{n_k},$$

where
$$n_1 + n_2 + \cdots + n_k = n,$$

then by § 187, we have

$$\begin{aligned}\frac{f(x)}{g(x)} = {} & \frac{A_1}{(a_1x + b_1)^{n_1}} + \frac{A_2}{(a_1x + b_1)^{n_1 - 1}} + \cdots + \frac{A_{n_1}}{a_1x + b_1} \\ & + \frac{B_1}{(a_2x + b_2)^{n_2}} + \frac{B_2}{(a_2x + b_2)^{n_2 - 1}} + \cdots + \frac{B_{n_2}}{a_2x + b_2} \\ & + \cdots\cdots\cdots\cdots \\ & + \cdots\cdots\cdots\cdots \\ & + \frac{L_1}{(a_kx + b_k)^{n_k}} + \frac{L_2}{(a_kx + b_k)^{n_k - 1}} + \cdots + \frac{L_{n_k}}{a_kx + b_k}, \qquad (1)\end{aligned}$$

since the fractions in the right member represent all the types of partial fractions mentioned in the theorem.

If we clear (1) of fractions, the left member of the resulting equation will be of degree less than n and the right member will be of degree $n - 1$. Therefore when we equate the coefficients of like powers of x in this equation in accordance with the principles already explained we shall get n linear equations in the n unknowns

$$A_1, A_2, \cdots, A_{n_1};\ B_1, B_2, \cdots, B_{n_2};\ \cdots;\ L_1, L_2, \cdots, L_{n_k}.$$

Moreover these equations must be solvable, since if they were not, the decomposition mentioned in the theorem of § 187 would be impossible.

If therefore we solve them and substitute the values of the unknowns in (1), we shall have the desired decomposition of $\frac{f(x)}{g(x)}$.

EXERCISES

Decompose the following fractions into partial fractions:

1. $\frac{3x+1}{(x+1)^2}$.
2. $\frac{x^2-1}{x(x+1)^2}$.
3. $\frac{2x^2+3x-1}{x^2(2x+5)}$.
4. $\frac{6}{(x^2-1)(x+1)}$.
5. $\frac{3x+4}{(2x+3)^3}$.
6. $\frac{2}{(x-1)^3}$.
7. $\frac{x^3+x^2+x+2}{(4x^2+12x+9)(x+1)^2}$.
8. $\frac{(2x+6)^2}{(x+5)^3}$.
9. $\frac{3x^2-550}{(3x-5)^3(2x+1)}$.
10. $\frac{x^3+1}{(x^2-4)^2}$.
11. $\frac{x^2+2}{(x^2-1)^2}$.
12. $\frac{x}{(x+2)^3}$.
13. $\frac{x^4}{x^3+9x^2+27x+27}$.
14. $\frac{6x^2+2}{(x+5)^2(2x-3)}$.
15. $\frac{27x^2+36x+27}{(2x+3)^2(3x+2)}$.
16. $\frac{x+2}{49x^2+28x+4}$.
17. $\frac{1}{(x+1)^2(x-1)^3}$.
18. $\frac{4x+3}{(x+6)^2(x-6)}$.
19. $\frac{x+3}{(x^2-9)(x-3)}$.
20. $\frac{6}{(x^2+3x+2)^2}$.

190. If we apply the preceding methods to the decomposition of $\dfrac{20x+40}{(x^2+1)(x+3)}$, we get

$$\frac{20x+40}{(x^2+1)(x+3)} = \frac{1+7i}{x+i} + \frac{1-7i}{x-i} - \frac{2}{x+3}.$$

The appearance of imaginary coefficients in this decomposition can be avoided by not carrying the decomposition so far. Thus if we retrace our steps in part by expressing the sum of the first two fractions of this decomposition as a single fraction, we get

$$\frac{20x+40}{(x^2+1)(x+3)} = \frac{2x+14}{x^2+1} - \frac{2}{x+3}.$$

Here there are no imaginary coefficients in the final form and the denominators are of the first or second degree.

If the coefficients of the original fraction are real, the decomposition can always be stopped at a point at which the coefficients are real and the denominators are of the first or the second degree or powers of such expressions.

By § 148, we know that any polynomial, as $g(x)$, of degree n is the product of n factors. Moreover if the coefficients of $g(x)$ are real, the imaginary factors (that is, those with imaginary coefficients) occur in conjugate pairs (§ 151), if at all. The part of the decomposition that corresponds to such a pair of imaginary factors of the first degree will be of the form

$$\frac{A}{x-a-ib} + \frac{B}{x-a+ib}.$$

But this is equal to

$$\frac{(A+B)x+(-Aa-Ba+iAb-iBb)}{x^2-2ax+a^2+b^2}.$$

Then in the decomposition we can put the single fraction

$$\frac{A_1x+B_1}{x^2-2ax+a^2+b^2},$$

where $A_1 = A+B$ and $B_1 = -Aa-Ba+iAb-iBb$.

It can be shown in general that if n_1 is the highest power of $x-a-ib$ and of $x-a+ib$ by which the denominator is divisible, then the sum of the partial fractions whose denominators are the different powers of $x-a-ib$ and of $x-a+ib$ is equal to the sum of fractions of the form

$$\frac{A_{n_1}x+B_{n_1}}{(x^2-2ax+a^2+b^2)^{n_1}},\quad \frac{A_{n_1-1}x+B_{n_1-1}}{(x^2-2ax+a^2+b^2)^{n_1-1}},\ \cdots,$$

$$\frac{A_1x+B_1}{x^2-2ax+a^2+b^2},$$

where the A's and B's do not contain x.

Then in the original decomposition in place of a pair of fractions whose denominators are conjugate imaginaries we can place a single fraction whose numerator is of the first degree with undetermined coefficients and whose denominator is a real factor of $g(x)$ of the second degree or a power of such a factor. The resulting equations for determining the undetermined constants in the numerators of the partial fractions will have only real coefficients and will be of the first degree. Hence these undetermined constants must have real values.

This partial decomposition of a rational fraction has therefore the advantage of involving only real coefficients, and it is sufficient for the uses to which decomposition into partial fractions is put.

Example 1.—Decompose $\dfrac{20x+40}{(x^2+1)(x+3)}$ into real partial fractions.

Put $$\frac{20x+40}{(x^2+1)(x+3)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+3}.$$

Clearing of fractions we get

$$20x+40=(Ax+B)(x+3)+C(x^2+1).$$

Equating coefficients of like powers of x,

$$0=A+C,$$
$$20=3A+B,$$
$$40=3B+C.$$

Whence $A = 2$, $B = 14$, $C = -2$, and the decomposition is

$$\frac{20x + 40}{(x^2 + 1)(x + 3)} = \frac{2x + 14}{x^2 + 1} - \frac{2}{x + 3}.$$

Example 2.— Decompose $\dfrac{3x - 5}{(x^2 + x + 1)^2(x + 1)}$ into real partial fractions.

$$\frac{3x - 5}{(x^2 + x + 1)^2(x + 1)} = \frac{Ax + B}{(x^2 + x + 1)^2} + \frac{Cx + D}{x^2 + x + 1} + \frac{E}{x + 1}.$$

Clearing of fractions,

$$3x - 5 = (Ax + B)(x + 1) + (Cx + D)(x^2 + x + 1)(x + 1) + E(x^2 + x + 1)^2.$$

Equating the coefficients of like powers of x,

$$\begin{aligned} C + E &= 0, \\ 2C + D + 2E &= 0, \\ A + 2C + 2D + 3E &= 0, \\ A + B + C + 2D + 2E &= 3, \\ B + D + E &= -5. \end{aligned}$$

Whence $A = 8$, $B = 3$, $C = 8$, $D = 0$, $E = -8$; and the decomposition is

$$\frac{3x - 5}{(x^2 + x + 1)^2(x + 1)} = \frac{8x + 3}{(x^2 + x + 1)^2} + \frac{8x}{x^2 + x + 1} - \frac{8}{x + 1}.$$

EXERCISES

Decompose the following fractions into real partial fractions:

1. $\dfrac{2x^2 - 5}{x^3 + 5x}$.

2. $\dfrac{3x - 2}{x^4 - 16}$.

3. $\dfrac{x^3 - 5x^2 + 2x + 1}{x^4 + 5x^2 + 4}$.

4. $\dfrac{x^3 + 1}{x^3 + x^2 + x}$.

5. $\dfrac{x^3 + 7}{(x^2 + 2x + 5)(x - 1)^2}$.

6. $\dfrac{4x^2 + x - 6}{(x^2 + 3)^2}$.

7. $\dfrac{6}{x^6 - 1}$.

8. $\dfrac{5}{(x^2 + 1)^2}$.

9. $\dfrac{11x^2+11}{(2x+3)(x^2+x+3)}$.

10. $\dfrac{x^2+2x+3}{x^3-1}$.

11. $\dfrac{x^2+4}{x^3+8}$.

12. $\dfrac{3x-1}{(x^3+1)^2}$.

13. $\dfrac{4x+5}{x^4-x^3+x^2}$.

14. $\dfrac{4}{y^4-81}$.

15. $\dfrac{x^2+x+2}{x^3+x^2+x+1}$.

16. $\dfrac{4x^3+4x^2+8x+6}{x^4+2x^2}$.

17. $\dfrac{1}{y^6-64}$.

18. $\dfrac{-2x^2-9x-10}{x^3+6x^2+5x}$.

19. $\dfrac{2x^2+14}{x^4+6x^2+8}$.

20. $\dfrac{3x+2}{x^6-2x^3+1}$.

CHAPTER XVII

LOGARITHMS

191. Definition of irrational exponents. — In § 58 it was stated that irrational exponents could be used in a way consistent with the five fundamental laws of exponents. This statement is based upon the meaning we attach to these exponents. We cannot give a complete justification of it here, but shall indicate briefly what this meaning is.

In the first place the student should recall that a decimal fraction can always be written as a common fraction, and that accordingly a number with a decimal exponent can be expressed in a way with which he is familiar. For example,

$$a^{.125} = a^{\frac{1}{8}} = \sqrt[8]{a}; \quad a^{2.33} = a^{\frac{233}{100}} = \sqrt[100]{a^{233}}.$$

Now, as he can readily verify, the numbers

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \cdots,$$

which are obtained in the process of finding the approximate square root of 2, are such that their successive squares are closer and closer to 2. As a matter of fact, it can be shown that when this sequence is continued indefinitely the successive numbers approach a certain definite limit, which is the positive square root of 2. (See § 202 for the definition of a limit.)

It can be shown further that the numbers of the corresponding sequence

$$a^1,\ a^{1.4},\ a^{1.41},\ a^{1.414},\ a^{1.4142},\ \cdots,$$

also approach a limit. It is this limit that we denote by the symbol $a^{\sqrt{2}}$.

In general, every irrational number n is the limit of a sequence of rational numbers

$$n_1, n_2, n_3, \cdots,$$

and the numbers of the sequence

$$a^{n_1}, a^{n_2}, a^{n_3}, \cdots$$

approach a limit, which we indicate by the symbol a^n.

192. Definition. *If $a > 0$ and $a^x = b$, x is said to be the logarithm of b, and b the antilogarithm of x, to the base a.*

The symbol for the logarithm of b to the base a is $\log_a b$.

Thus, $2^5 = 32$, and therefore 5 is the logarithm of 32, and 32 the antilogarithm of 5, to the base 2, or $5 = \log_2 32$.

EXERCISES

1. $\log_3 9 = ?$ $\log_9 3 = ?$ $\log_5 1 = ?$ $\log_a a = ?$ $\log_a a^2 = ?$

2. Complete the following table:

Number	Base	Logarithm
243	3	
2	8	
	10	3
	5	4
2	16	
343		3
3		$\frac{1}{3}$
	$\frac{1}{2}$	3

193. We can take for the base a any positive number except 1.

Theorem. — *For any positive number b and any positive number a, except 1, there is a real number x such that*

$$a^x = b.$$

In other words, *every positive number has a real logarithm with respect to any positive base except 1.*

The proof of this theorem is too difficult for this book, and therefore is omitted.

Negative numbers have no real logarithms. It can be shown that the logarithm of a negative number to a positive base is imaginary. We shall confine our attention here to real logarithms.

194. The logarithms of all positive numbers with respect to any given positive base different from 1 form a system of real numbers that possess the following properties:

I. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of these numbers.*

For, if $\log_a b = x, \log_a c = y,$

or $a^x = b$ and $a^y = c,$

then by multiplication

$$a^{x+y} = bc.$$

But this is only another way of saying

$$\log_a bc = x + y = \log_a b + \log_a c.$$

This proves the theorem as far as concerns the product of two numbers.

The completion of the proof is left to the student.

II. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For, if $\log_a b = x, \log_a c = y,$

or $a^x = b, a^y = c,$

then by division, $a^{x-y} = \frac{b}{c}.$

But this is only another way of saying

$$\log_a \frac{b}{c} = x - y = \log_a b - \log_a c.$$

III. *If m is any real number, the logarithm of b^m is equal to m times the logarithm of b.*

For, if $\log_a b = x,$

or $a^x = b,$

then we have, by equating the mth powers of the two members of this equation,

$$a^{mx} = b^m.$$

Hence $\log_a b^m = mx = m \cdot \log_a b.$

If in III m is equal to $\frac{1}{n}$, where n is an integer, we have,

the logarithm of the positive nth root of a positive number is equal to the logarithm of the number divided by n.

EXERCISES

Given $\log_{10} 2 = .3010$, $\log_{10} 3 = .4771$, $\log_{10} 7 = .8451$
$\log_3 7 = 1.7712$, $\log_3 11 = 2.1827$; find:

1. $\log_{10} 14$.

HINT. — $14 = 2 \cdot 7$. Hence $\log_{10} 14 = \log_{10} 2 + \log_{10} 7$.

2. $\log_3 \frac{11}{7}$.
3. $\log_{10} \sqrt[3]{42}$.
4. $\log_3 \frac{7}{11}$.
5. $\log_{10} \sqrt{21}$.
6. $\log_3 \sqrt[3]{5929}$.
7. $\log_{10} \sqrt[5]{2^3 \cdot 7^4}$.
8. $\log_{10} 2058$.
9. $\log_3 \sqrt{7}$.
10. $\log_3 \sqrt[3]{77}$.

195. The systems of logarithms most frequently used. — For reasons that are explained in the Differential Calculus it is convenient for certain purposes to use as base of the system of logarithms an irrational number whose approximate value is 2.71828. Logarithms to this base are called **natural** logarithms. On the other hand, for purposes of numerical computation, there is an advantage in using the base 10. The reason for this will be explained in § 198, I. Logarithms to the base 10 are called **common** logarithms.

196. Change of base.—Since different systems of logarithms are in use, it is important to know how to change from one system to another. The following theorem explains how this can be done.

Theorem. — *The logarithm of a number to the base b is equal to the product of its logarithm to the base a and the logarithm of a to the base b.*

If $\log_a x = u$ and $\log_b x = v$; that is,

if $a^u = x$

and $b^v = x$,

then $a^u = b^v$.

Hence $a = b^{\frac{v}{u}}$,

or $\frac{v}{u} = \log_b a$.

Hence $v = u \cdot \log_b a$,

or $\log_b x = \log_a x \cdot \log_b a$.

This proves the theorem.

The following theorem enables us to express this relation in another form, which is sometimes useful.

Theorem. — *The logarithm of a to the base b is the reciprocal of the logarithm of b to the base a.*

For, if $\log_b a = x$, or $a = b^x$;

then $a^{\frac{1}{x}} = b$, or $\log_a b = \frac{1}{x}$.

Hence $\log_b a = \frac{1}{\log_a b}$.

The relation proved in the preceding theorem can now be written thus:

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

197. Common logarithms. — We shall confine our attention in the rest of this chapter to common logarithms and it will therefore be unnecessary to indicate the base in the

symbol for a logarithm. Thus we shall write $\log 2$ instead of $\log_{10} 2$.

Now it is easy to see what the logarithm of 100, or of 1000, is; but it is not easy to determine the logarithms of most numbers, say 20, for example. The approximate value of the logarithm of such a number (that is, of a number that is not an integral power of 10) can be determined only by a series of computations that are too long and complicated to be explained here.

The integral part of a logarithm is called the **characteristic** of the logarithm and can be determined by inspection in a way that we shall explain (§ 198). The fractional part of the logarithm is called the **mantissa** and is given in a table. Thus, the approximate value of the logarithm of 20 being 1.3010, the characteristic is 1 and the mantissa .3010.

By virtue of their properties given in § 194, logarithms are extremely useful in shortening numerical computation, and it is for this reason we are considering them here.

Suppose, for example, we want the fifth root of 20. We have already seen that

$$\log 20 = 1.3010.$$

But by § 194, III, $\log \sqrt[5]{20} = \frac{1}{5} \log 20.$

Hence $\log \sqrt[5]{20} = .2602.$

Now the number whose logarithm is .2602 can be determined approximately from the table in a way to be explained in § 198, II. And when this is determined we shall have the approximate value of $\sqrt[5]{20}$.

The student would appreciate the advantage in this use of logarithms if he were to attempt to find the fifth root of 20 directly.

The operations of multiplication, division, and raising to powers can also often be shortened by the use of logarithms.

But for this numerical work it is necessary to use a table of logarithms.

198. Use of a logarithmic table. — We proceed therefore to explain the two ways in which the table can be used. The procedure is based upon the fact that if a and b are greater than 1 and b is greater than a then $\log b > \log a$.

I. *To find the logarithm of a number.* — The table in this book is so arranged that the mantissa of the logarithm of a number of three digits can be taken out directly. To do this we look in the first column for the first two digits (counting from the left) and then over on this row to the column headed by the right-hand digit. The number in this row and column is the mantissa we are looking for. It is understood that all the numbers in the body of the table are to be preceded by the decimal point. Thus, the mantissa of the logarithm of 483 is .6839.

The characteristic is determined without the use of the table from the following considerations:

$$10^0 = 1, \text{ or } \log 1 = 0.$$
$$10^1 = 10, \text{ or } \log 10 = 1.$$
$$10^2 = 100, \text{ or } \log 100 = 2.$$
$$10^3 = 1000, \text{ or } \log 1000 = 3.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence the logarithm of any number between 1 and 10 lies between 0 and 1, and has therefore the characteristic 0. Moreover every such number (when written as an integer or a decimal) has one digit to the left of the decimal point. The logarithm of any number between 10 and 100 is between 1 and 2 and has the characteristic 1. Such a number has two digits to the left of the decimal point.

In general, *since* 10^n *is the smallest integer with* $n+1$ *digits the characteristic of the logarithm of a number greater than* 1 *is one less than the number of digits to the left of the decimal point in the number.*

We agree to say that the numbers 2015, 201,500, 2.015, and .0002015 have the same sequence of digits. In general, two numbers that have the same digits in the same order and differ only in the position of the decimal point and in the ciphers that may be necessary to indicate the position of the decimal point are said to have the *same sequence of digits.*

The position of the decimal point in a number can be changed any number of places to the right or left by multiplying or dividing the number by some integral power of 10. Moreover the logarithm of any integral power of 10 is an integer.

For example, $\log 432 = 2.6355$;

that is, $432 = 10^{2.6355}$.

Dividing each member of this equation by 10, we get

$$43.2 = 10^{1.6355}.$$

Hence, $\log 43.2 = 1.6355$.

In a similar way we see that

$$\log 4.32 = 0.6355,$$
$$\log .432 = .6355 - 1,$$
$$\log .0432 = .6355 - 2,$$
$$\log .00432 = .6355 - 3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

From this we see that, for example,

$$\log .432 = .6355 - 1 = -.3645.$$

But a logarithm is usually not written in this way, but thus, $9.6355 - 10$.

Similarly, we say

$$\log .0432 = 8.6355 - 10,$$
$$\log .00432 = 7.6355 - 10.$$

This agreement always to write the logarithm in such a way as to have its mantissa positive is based on grounds of convenience in the use of the tables.

If there are n ciphers immediately following the decimal point in a number less than 1, the characteristic will be taken as $-n-1$. It is convenient to write this $9 - n - 10$.

If we agree to write the logarithms of numbers less than 1 in the way indicated here, we can make the following statements:

The mantissa of the logarithm of a number is always positive.

The logarithms of two numbers that have the same sequence of digits have the same mantissa and differ only in their characteristics.

The characteristic of the logarithm of a number less than 1 *is equal to* $9 - n - 10$, *where* n *is the number of ciphers immediately following the decimal point in the number. This characteristic is written in two parts. The first part,* $9 - n$, *is written at the left of the mantissa and the* -10 *at the right.*

Thus $\log .00432 = 7.6355 - 10.$

Here $n = 2$ and $9 - n = 7.$

It is clear from what has been said that in looking in the table for the mantissa of the logarithm of a number we do not need to pay any attention to the position of the decimal point in the number, — the characteristic is the only thing about the logarithm that is affected by the position of the decimal point.

The finding of the logarithm of a number of more than three digits from this table is not so simple. The method of procedure is best illustrated by an example.

Find log 5421.

From the table $\log 5420 = 3.7340,$

$\log 5430 = 3.7348.$

These two logarithms differ by .0008 and correspond to numbers that differ by 10. Now the numbers 5420 and 5421 differ by 1 and we assume that the difference in their logarithms is .1 of the difference in the logarithms of 5420 and 5430; that is, .1 of .0008. This is .00008. Hence

$$\log 5421 = 3.7340 + .00008 = 3.7341.$$

If we were looking for the logarithm of 5427, we should add .7 of .0008 to the logarithm of 5420 since the difference be-

tween 5427 and 5420 is .7 of the difference between 5430 and 5420.

In doing this work it is customary, in the interests of brevity, to omit the decimal point in giving the difference between the logarithms of two numbers. Thus, in the example just given, we say that the difference between the logarithms of 5420 and 5430 is 8, instead of .0008.

We have assumed that the difference between the logarithms of two numbers is proportional to the difference between the numbers. This is not strictly true. However, when the difference between the numbers is small, as it is in these cases, the error due to this assumption is very slight and can safely be neglected, since most of the logarithms given in the table are themselves only approximations.

In using four-place tables the student should keep the antilogarithms in all his calculations down to four significant figures. The point of this remark is illustrated by the fact that 43.526 and 43.53 have the same four-place logarithm.

From the table $\log 43.600 = 1.6395$

$\log 43.500 = 1.6385$

Difference in logs. $= 10$

The difference between 43.500 and 43.526 is .26 of the difference between 43.500 and 43.600, and .26 of the difference between the logarithms of these last two numbers is 3. Hence

$$\log 43.526 = 1.6388.$$

On the other hand the difference between 43.500 and 43.53 is .3 of the difference between 43.500 and 43.600, and .3 of the difference between the logarithms of these numbers is 3. Hence

$$\log 43.53 = 1.6388.$$

When the digits to the right of the fourth place (counting from the left) are dropped off, increase the digit in the fourth place by 1 if the digit in the fifth place is greater than 5, or if the fourth digit is odd and the fifth one is equal to 5 ; in other cases leave the fourth one unchanged.

EXERCISES

Find the logarithms of the following numbers:

1. 461.	8. 24.86.	15. 3.
2. .0024.	9. 2562.	16. 12.
3. 500000.	10. 14380.	17. 58.64.
4. .000005.	11. .06473.	18. 5437.
5. 1934.	12. 374.6.	19. .6892.
6. 7.832.	13. 8001.	20. 40.15.
7. .6294.	14. 19.03.	

II. *To find the antilogarithm of a logarithm.* — If the mantissa of the given logarithm is found in the table, it is an easy matter to find the antilogarithm. Suppose, for example, we wish to find the antilogarithm of $7.9258 - 10$.

Now .9258 appears in the table as the mantissa of the logarithms of those numbers whose sequence of digits is 843. Since the characteristic is $7 - 10$, the antilogarithm must be less than 1 and there must be two ciphers immediately following the decimal point.

Hence $\text{antilog}\,(7.9258 - 10) = .00843,$

or $\log .00843 = 7.9258 - 10.$

When the given mantissa is not found in the table the matter is not so simple. The method of procedure in such cases is illustrated by the following example:

Find antilog 2.6959.

In this part of the tables when a four-place number is increased by 10, the mantissa of its logarithm is increased by 9. Hence we assume that if a mantissa is increased by 1, the number it corresponds to is increased by $\frac{10}{9}$. Since the mantissa 6959 is obtained by increasing the mantissa 6955 by 4, the antilogarithm of 6959 should be that of 6955 increased by $4 \times \frac{10}{9} = 4$. Hence the given mantissa must correspond to numbers whose sequence of digits is 4964. Since the charac-

teristic is 2, the antilogarithm must be greater than 1, and it must have three digits to the left of the decimal point.

Hence $\text{antilog } 2.6959 = 496.4,$

or $\log 496.4 = 2.6959.$

The difference between the mantissæ of the logarithms of two consecutive numbers is called a **tabular difference.**

The preceding example suggests the following rule for finding the antilogarithm of a logarithm whose mantissa does not appear in the table.

Rule. — *Find two consecutive mantissæ in the table between which the given mantissa lies, and get the difference between them. Multiply the difference between the smaller of these mantissæ and the given mantissa by* 10 *and divide the product by the tabular difference just found. Annex the quotient, expressed in the nearest integer, to the sequence of three digits corresponding to the smaller mantissa of the table.*

In the resulting sequence of digits place the decimal point as indicated by the given characteristic.

EXERCISES

Find the antilogarithms of the following:

1. 2.6542.
2. 9.9805 − 10.
3. 1.8457.
4. 8.1644 − 10.
5. 7.8162 − 10.
6. 1.6245.
7. .1287.
8. 9.1287 − 10.
9. 1.1287.
10. 8.9970 − 10.
11. 2.0312.
12. 3.
13. 8.
14. .8354.
15. .1870.
16. 1.8125.
17. 9.8449 − 10.
18. 7.3950 − 10.
19. 2.6045.
20. 8.4857 − 10.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

199. Numerical computation by means of logarithms. — It is of great importance for the student to be able to perform fairly complicated numerical computations with accuracy. He must not think that a numerical error is of small consequence so long as his work has been done on the correct principle. In all but the simplest computations he will find it worth while, in the interests of accuracy, to give attention to the proper arrangement of his work.

The following general arrangement is suggested:

Find the value of $\frac{5.25 \times 89.46}{17.35 \times 38.61}$.

$$\begin{array}{r} \log 5.25 = 0.7202 \\ \log 89.46 = 1.9516 \\ \hline \log 5.25 \times 89.46 = 12.6718 - 10 \\ \log 17.35 \times 38.61 = \quad 2.8260 \\ \hline \log \dfrac{5.25 \times 89.46}{17.35 \times 38.61} = \quad 9.8458 - 10 \end{array} \qquad \begin{array}{r} \log 17.35 = 1.2393 \\ \log 38.61 = 1.5867 \\ \hline \log 17.35 \times 38.61 = 2.8260 \end{array}$$

$$\therefore \frac{5.25 \times 89.46}{17.35 \times 38.61} = .7012.$$

In this work we changed the form of $\log 5.25 \times 89.46$ by adding and subtracting 10 in order to avoid a negative form for the logarithm of the required quotient.

It is best to lay out all the work before looking up any logarithms.

Thus, in the preceding problem the work was arranged in the following way before any logarithms were filled in:

$$\begin{array}{r} \log 5.25 = 0. \\ \log 89.46 = 1. \\ \hline \log 5.25 \times 89.46 = \\ \log 17.35 \times 38.61 = \\ \hline \log \dfrac{5.25 \times 89.46}{17.35 \times 38.61} = \end{array} \qquad \begin{array}{r} \log 17.35 = 1. \\ \log 38.61 = 1. \\ \hline \log 17.35 \times 38.61 = \end{array}$$

$$\therefore \frac{5.25 \times 89.46}{17.35 \times 38.61} =$$

It is suggested that the student find the value of this fraction without the use of logarithms and compare his result with that obtained above.

The following example illustrates the use of an important artifice in logarithmic computation:

Find $\sqrt[3]{.05}$. $\log .05 = 8.6990 - 10.$

$\therefore \log \sqrt[3]{.05} = 2.8997 - \frac{10}{3}.$

Now it is inconvenient to have the fraction $\frac{10}{3}$ occurring here, and we can avoid it by writing

$$\log .05 = 28.6990 - 30.$$

Then $\log \sqrt[3]{.05} = 9.5663 - 10.$

The general procedure in such cases is to arrange the logarithms in such a way that the final logarithm will appear in the standard form.

The principal advantage in the use of logarithms for performing the operations of multiplication, division, and raising to an integral power depends upon one's facility and accuracy in the use of logarithmic tables. But in the extraction of a root there is a further important and obvious advantage. See for example the illustration on page 243.

EXERCISES AND PROBLEMS

Perform the following indicated operations by the use of logarithms:

1. $.372 \times 4.26 \times 37.1.$
2. $98.41 \times 39.72.$
3. $(9.23)^2(.761)^3.$
4. $1246 \times 923.4 \times .001672.$
5. $\sqrt[3]{2}.$
6. $9762 \div 48.75.$
7. $\dfrac{3762 \times 7931}{2945 \times 1976}.$
8. $\sqrt[3]{100}.$
9. $(6.21)^3(9.432)^5 \div (46.74)^4.$
10. $\sqrt[3]{\dfrac{3.96 \times 4.87}{13.9 \times (5.67)^2}}.$
11. $(.8762)^2(9.436)^2 \times (.7582)^5.$
12. $\sqrt[5]{6943}.$
13. $\dfrac{9842}{7963}.$
14. $\sqrt[4]{\dfrac{75.36}{4.932}}.$

15. $\dfrac{(12)^3(496)^2 \times 976.4}{5621 \times 498.6 \times 71.34}$.

16. $\sqrt[3]{\dfrac{19.3 \times (47.21)^2 (5.931)^5}{4500}}$.

17. $\sqrt[10]{10}$.

18. $\sqrt[7]{14}$.

19. $\sqrt[5]{20}$.

20. $\sqrt[5]{(468)^3}$.

21. What is the weight in tons of a marble sphere $2\frac{1}{2}$ feet in diameter if a cubic foot of water weighs 62.36 pounds and the specific gravity of marble is 2.7 ?

22. When a weight of m grams is attached to the free end of a suspended brass wire, the wire stretches S centimeters. If l is the length and r the radius of the wire in centimeters, $g = 980$, and $k = 1051 \cdot 10^7$, then

$$S = \frac{mgl}{\pi r^2 k}.$$

Compute S when $m = 750$, $l = 164$, and $r = .4$.

23. What will \$ 850 amount to in 10 years at 4 %, interest being compounded annually ?

This example affords a good illustration of the limitations of a four-place logarithmic table.

We have $A = 850\ (1.04)^{10}$, and therefore $\log A = \log 850 + 10 \log 1.04$. Now all that we can be sure of in a four-place table is that no logarithm can be more than .00005 too much or too little. Accordingly in this expression $\log 850$ is subject to an error not greater than .00005 and $10 \log 1.04$ to an error not greater than .0005, and therefore $\log A$ as computed in this way cannot be more than $.00005 + .0005 (= .0006)$ too much or too little. The computed value of $\log A$ is 3.0994 and hence its true value lies between 3.1000 and 3.0988. The true value of A therefore lies between \$ 1260 and \$ 1255, while the computed value is \$ 1257. A six-place table shows that the true value of A lies between \$ 1258.21 and \$ 1258.18. A seven-place table would give A correct to the nearest cent. In practice the nature of the problem determines the number of places that should be given in the table used. The greater the number of places given in the table the greater the accuracy of the computations.

24. What sum will amount to \$ 1500 in 5 years at 3%, interest being compounded annually ?

25. If a string weighing .0098 grams per centimeter is stretched between two bridges 60 centimeters apart by a weight of 10 grams, how many vibrations will it make per second? (See Ex. 1, p. 105.)

26. The weight w in grams of a cubic meter of aqueous vapor saturated at 15° is given by the formula

$$w = \frac{1293 \times 12.7 \times 5}{(1 + \frac{15}{273})760 \times 8}.$$

Compute w.

27. The volume v in litres of three kilograms of mercury at 85° is given by the formula

$$v = \frac{30}{13.6}\left(1 + \frac{85}{5550}\right).$$

Compute v.

28. The time t of oscillation of a pendulum of length l centimeters is given by the formula

$$t = \pi\sqrt{\frac{l}{980}}.$$

Find the time of oscillation of a pendulum 74.36 centimeters long.

29. What must be the length of a pendulum in order that its time of oscillation be one second?

30. The velocity v of a body that has fallen s feet is given by the formula

$$v = \sqrt{64.3\, s}.$$

What is the velocity acquired by a body falling 29 feet 7 inches?

200. Exponential equations. — *An exponential equation is an equation that involves the unknown in an exponent.*

The simplest exponential equations can be solved by inspection.

For example, if $4^x = 16$,
it is immediately obvious that $x = 2$.

Many exponential equations that cannot be solved by inspection can readily be solved by the use of logarithms in the way illustrated in the following example:

Solve $2^x = 5.$

Since the logarithms of equal numbers must be equal, we can equate the logarithms of the two members of this equation. In this way we get

$$\log 2^x = \log 5,$$

or

$$x \log 2 = \log 5.$$

Hence

$$x = \frac{\log 5}{\log 2} = \frac{.6990}{.3010} = 2.32.$$

NOTE. — The student should observe the difference between $\frac{\log 5}{\log 2}$ and $\log \frac{5}{2}$.

EXERCISES AND PROBLEMS

Solve the following equations:

1. $3^x = 4.$
2. $5^x = 125.$
3. $(1.03)^x = 2.$
4. $6^{x^2} = 21.$
5. $(1.02)^x = 5.$
6. $7^{2x+1} = 12.$

7. In how many years will a sum of money double itself at 3 %, interest being compounded annually?

In one year \$1 will amount to \$1.03; at the end of two years the amount will be $1.03 \times 1.03 = (1.03)^2$; and at the end of x years the amount of \$1 will be $(1.03)^x$. Hence x must be such that $(1.03)^x = 2$. (See Ex. 3.)

8. In how many years will a sum of money double itself at 3 %, interest being compounded semiannually?

9. In how many years will \$1000 amount to \$1338 at 6 %, interest being compounded annually?

If p dollars amount to a dollars in n years at r%, compounded annually, then

$$a = p\left(1 + \frac{r}{100}\right)^n.$$

10. Solve this equation for p, r, and n in turn.

When solving for n, and for r when $n > 2$, it is best to use logarithms.

11. What must be the rate of interest in order that \$2500 may amount to \$2814.00 in four years, interest being compounded annually?

CHAPTER XVIII

VARIATION

201. Constants and variables. — Many of the letters that we use in mathematics represent only one number throughout a given discussion. They are called **constants.** A letter, on the other hand, that represents different numbers in one discussion is called a **variable.**

Thus, if t represents the time that has elapsed since a given moment, it is a variable, since this time is changing. Also in the equation $y = x^2 + 6x + 5$ the letters x and y are variables, since they represent any two numbers that are related to each other in the way described by the equation.

In some problems two or more variables are involved, and in such cases there is usually a connection between the variables. The particular form of this connection depends upon the nature of the problem, but there are certain forms that occur much more frequently than others. These are described in the following paragraphs.

If two variables are so related that their ratio is constant, either one is said to *vary directly as the other.*

Sometimes the word "directly" is omitted, and we say that one of the variables varies as the other. The two expressions have the same meaning.

If a body is moving at a uniform rate, the distance passed over varies as the time, since $\frac{d}{t} =$ a constant; namely, the rate.

The symbol $\propto$ when placed between two numbers indicates that one varies as the other. Thus, $d \propto t$ means that d *varies as* t, or that $\frac{d}{t} = k$, where k is a constant.

If two variables are so related that their product is constant, either one is said to *vary inversely as the other.* If x varies inversely as y, it varies directly as the reciprocal of y, since if $xy = k$, then $\frac{x}{\frac{1}{y}} = k$, or $x = \frac{k}{y}$. We can accordingly express that x varies inversely as y by the symbol $x \propto \frac{1}{y}$.

If a body moving at a uniform rate of r feet per minute goes a distance of d feet in t minutes, then $d = rt$. In order to get over the distance of d feet in a different time the rate must be changed in such a way that the product of the new rate and the new time still equals d. Hence, under these circumstances, the rate varies inversely as the time.

If three variables are so related that the ratio of the first one to the product of the other two is constant, the first one is said to *vary jointly as the other two.* Thus, if x varies jointly as y and z, then $\frac{x}{yz} = k$, or $x = kyz$, where k is a constant.

The first variable is said to vary directly as the second one and inversely as the third if it is equal to the product of a constant and the ratio of the other two. Thus, x varies directly as y and inversely as z if $x = \frac{ky}{z}$, where k is a constant.

If the first variable is equal to the product of a constant and the ratio of the second to the square of the third, it is said to *vary directly as the second and inversely as the square of the third.* In this case $x = \frac{ky}{z^2}$. Thus, the attractive force that draws two particles of matter together varies directly as the product of their masses and inversely as the square of the distance between them.

It is common in physics to meet with two variables so related that one varies inversely as the square of the other. In this case $x = \frac{k}{y^2}$.

Example. — If $x \propto y$, what is the value of x when $y = 20$?

We are not given enough information about the relation between x and y to enable us to answer this question, for all we are told is that

$$x = ky,$$

where k is some constant whose value is not given.

But if we are given the additional information that $x = 10$ when $y = 8$, we can find the value of k, and then answer the question.

Since $x = ky$,

and $x = 10$ when $y = 8$, we have

$$10 = 8k,$$

and $$k = \tfrac{5}{4}.$$

Hence $$x = \tfrac{5}{4}y,$$

and therefore $x = 25$ when $y = 20$.

PROBLEMS

1. If x varies inversely as y and $x = 4$ when $y = 9$, what is the value of x when $y = 15$?

2. If x varies jointly as y and z and $x = 7$ when $y = 5$ and $z = 12$, what is the value of x when $y = 14$ and $z = 22$?

3. If $x \propto \frac{y}{z^2}$, and $x = 15$ when $y = 4$ and $z = 14$, what is the value of x when $y = 20$ and $z = 18$?

4. If x varies inversely as the square of y, what will be the effect on y of doubling the value of x?

5. The distance passed over by a body falling from rest varies directly as the square of the number of seconds during which it has fallen, and in 3 seconds it falls approximately 144.9 feet. How far will it fall in 5 seconds?

6. The velocity acquired by a body falling from rest varies directly as the number of seconds during which it has fallen, and its velocity at the end of 2 seconds is 64.4 feet per second. What is its velocity at the end of 6 seconds?

7. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. If an object weighs 50 pounds at the sea level, what would be its weight on top of a mountain a mile high? Assume that the radius of the earth is 4000 miles.

8. When an object is taken below the surface of the earth its weight varies directly as its distance from the center of the earth. If an object weighs 200 pounds at the surface, how much would it weigh midway between the center and the surface? How much would it weigh at the center?

9. Would an object that weighs 100 pounds at the surface of the earth weigh more 5 miles above the surface than it does 5 miles below the surface, or less?

10. The time of vibration of a simple pendulum varies directly as the square root of its length. If the time of vibration of a pendulum 39.14 inches long is one second, how long must a pendulum be in order that its time of vibration shall be two seconds?

11. A pendulum two meters long makes 61650 vibrations in a day. Find the length of a pendulum whose time of vibration is one second?

12. Represent graphically the relation between two variables when one varies directly as the other.

13. Represent graphically the relation between two variables when one varies inversely as the other.

14. Represent graphically the relation between two variables when one varies directly as the square of the other.

15. The amount of light received on a page from a given source varies directly as the size of the page and inversely as the square of its distance from the source of the light. One page is twice as large as another one and twice as far from the source of the light. Which page receives the more light?

16. A metallic sphere whose radius is 3 inches weighs 32 pounds. How much will a sphere of the same material whose radius is 5 inches weigh, if the volume of a sphere varies directly as the cube of its radius?

CHAPTER XIX

INFINITE SERIES

202. Limit of a variable. — Consider the sum of the first n terms of the geometric progression whose first term is 1 and whose common ratio is $\frac{1}{2}$.

$$s_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - (\tfrac{1}{2})^{n-1}.$$

The value of s_n depends upon the number of terms represented by it; that is, upon the value of n. Hence if we give different values to n, s_n becomes a variable.

If we fix in mind any small positive number, say $\frac{1}{1000000}$, an easy computation will show that s_n differs from 2 by less than this number if $n = 21$, and the difference will also be less than $\frac{1}{1000000}$ for all greater values of n. It is immaterial how small the positive number is that we have in mind. As soon as we fix upon it, we can take enough terms of the series so that the sum of these terms or of any greater number differs from 2 by less than this number.

We express these facts by saying that 2 is the limit of s_n as n increases without limit.

The relation between s_n and 2 can be exhibited graphically by considering the successive values of s_n as abscissæ of points. Of course we cannot mark all of these points, since there is an unlimited number of them. But a few of the first ones are sufficient to indicate that they are getting successively nearer and nearer to 2.

In general, we say that a variable x approaches a constant k as a **limit** if its law of variation is such that, when we fix in

mind any positive number whatever, the difference between x and k will become and remain less in absolute value than this number.

The student should observe the importance of the word *remain* in this definition.

In the preceding illustration, for example, the absolute value of the difference between s_n and $1\frac{3}{3}\frac{1}{2}$ becomes less than any positive number we can think of. As a matter of fact, it actually becomes 0 when $n = 6$. But when n increases from 6 the absolute value of this difference increases and cannot for any following value of x be less than $\frac{1}{6}\frac{1}{4}$. We therefore do not say that s_n approaches $1\frac{3}{3}\frac{1}{2}$ as a limit.

It follows from what has just been said that a variable can have only one limit.

The student should note that the limit of a variable is always a constant.

In the illustration the variable is increasing and is therefore always less than its limit. But some variables are decreasing and therefore always greater than their limits.

If, for example, S represents the area of a regular polygon circumscribed about a circle, and if we increase without limit the number of sides of the polygon, S is a variable which decreases toward the area of the circle as its limit.

Still other variables are increasing part of the time and decreasing part of the time.

For example, the sum s_n of the first n terms of the geometric progression whose first term is 1 and whose common ratio is $-\frac{1}{2}$.

$$s_n = \frac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} = \frac{2}{3} - \frac{2}{3}(-\frac{1}{2})^n.$$

Here the limit is $\frac{2}{3}$, and s_n is alternately greater than and less than the limit.

Some variables do not approach any limit, as, for example, the variable t described in § 201.

203. Convergent and divergent series. — A series with an unlimited number of terms is called an **infinite series.**

If the sum of the first n terms of an infinite series approaches a limit as n increases without limit, the series is said to be **convergent.**

The geometric progression referred to in § 202 is a convergent series, and in general any geometric progression with an unlimited number of terms and a common ratio greater than -1 and less than 1 is a convergent series (see § 100). But not every convergent series is a geometric progression, as we shall see in § 206.

If as n increases without limit, s_n does not approach a limit, the series is said to be **divergent.**

Example 1. — Consider the series

$$1 + 2 + 3 + \cdots + n + \cdots.$$

If we think of any positive number M, no matter how large, $s_n > M$ for all values of n that are greater than M. Hence s_n does not approach a limit and the series is divergent.

Example 2. — Consider the series

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots.$$

Here s_n is alternately 1 and 0 for successive values of n and therefore does not approach a limit. Hence the series is divergent.

There is an important difference between these two series. In the first one s_n can be made as great as we please by taking n sufficiently great, while in the second one s_n never exceeds 1.

204. The general term of a series. — An infinite series is not fully described until we are told how to form any given term. This information is usually supplied by means of the nth, or general, term. It is important therefore that the student should be able to write down any term of the series when the general term is given.

Example. — Write down the first five terms of the series whose nth term is $\dfrac{1}{2n(2n-1)}$. These terms are:

$$\frac{1}{2 \cdot 1}, \quad \frac{1}{4 \cdot 3}, \quad \frac{1}{6 \cdot 5}, \quad \frac{1}{8 \cdot 7}, \text{ and } \frac{1}{10 \cdot 9}.$$

EXERCISES

Write down the first three terms and the $(n+1)$th term of the series whose nth term is:

1. $\frac{1}{2n}$.
2. $\frac{1}{n^3}$.
3. $\frac{1}{n(n+1)}$.
4. $\frac{1}{n!}$.
5. $\frac{n}{3^n}$.
6. $\frac{1}{2n-1}$.
7. $\frac{x^n}{n!}$.
8. $\frac{1}{n}\frac{(x-2)^n}{2^n}$.
9. $(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$.
10. $(-1)^{n+1}\frac{x^{2n}}{(2n)!}$.

SERIES ALL OF WHOSE TERMS ARE POSITIVE

205. The only problem in connection with infinite series that will be considered in this book is that of determining whether a given series is convergent or divergent. This is an extremely difficult problem, and we shall discuss only the simpler phases of it.

We shall suppose at first that the series we are dealing with have all their terms positive, and we shall make use of the following

Fundamental principle. — *If a variable always increases (or at least never decreases) and never gets greater than a given number M, then it approaches a limit which is either M or less than M.*

206. Tests for convergence and divergence. — The following four tests for series with positive terms are the simplest and most important ones.

I. Comparison test for convergence. — Let

$$a_1 + a_2 + \cdots + a_n + \cdots$$

be an infinite series with positive terms. If a second infinite series with positive terms

$$b_1 + b_2 + \cdots + b_n + \cdots$$

is convergent, and if every term of this latter series is greater than, or equal to, the corresponding term of the first series, then the first series is convergent.

Denote by s_n and S_n the sums of the first n terms of these two series respectively. As n increases without limit S_n, by hypothesis, approaches a limit. Call this limit B.

Since the series have positive terms, s_n increases as n increases, but for no value of n is it greater than S_n, which in turn is always less than B. Hence $s_n < B$ for all values of n, and we know from the *fundamental principle* that it approaches a limit. The series is therefore convergent.

Example 1. — Consider the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots. \tag{1}$$

We know that the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots \tag{2}$$

is convergent. The first two terms of series (1) are the same as the first two terms of series (2), but the third term of series (1) is less than the third term of series (2) since $\frac{1}{6} < \frac{1}{4}$. In general, the nth term of series (1) is less than the nth term of series (2) when $n > 2$, since,

$$\frac{1}{1 \cdot 2 \cdot 3 \cdots n} < \underbrace{\frac{1}{2 \cdot 2 \cdots 2}}_{(n-1) \text{ factors}}.$$

Hence series (1) is convergent.

Example 2. Consider the series

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \cdots \tag{3}$$

After its first term this series is the same as series (1). Hence if s_n and S_n denote the sums of the first n terms of series (1) and series (3) respectively, we have

$$S_n = 1 + s_{n-1}.$$

Now we saw in Ex. 1 that as n increases without limit, s_n, and therefore also s_{n-1}, approaches a limit. Hence S_n approaches a limit and series (3) is convergent.

In considering the convergence of a series it is frequently desirable, as in Ex. 2, to investigate the new series obtained by dropping off the first few terms of the given series. Suppose that we drop off the first r terms and that the sum of these terms is s. Then if S_n and s_n denote the sums of the first n terms of the given series and the new series respectively, we have $S_n = s + s_{n-r}$, for values of $n > r$.

Now s is a constant, and therefore either one of the variables S_n and s_{n-r} approaches a limit if the other one does. Hence *the two series are either both convergent or both divergent.*

The student is reminded that s_n and s_{n-r}, where r is a constant, represent different stages of the same variable.

The conclusion just reached applies also to series some of whose terms are negative.

In order to use Test I we must know some convergent series with positive terms in order to have a basis for comparison; and the more of these we know the wider the range of usefulness of this test will be.

EXERCISES

Show that the following series are convergent:

1. $1 + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} + \cdots + \frac{1}{(n!)^2} + \cdots.$

2. $\frac{1}{3} + \frac{1}{3^4} + \cdots + \frac{1}{3^{n^2}} + \cdots.$

3. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots.$

4. $\frac{1}{2} + \frac{1}{1+2^2} + \frac{1}{1+3^3} + \cdots + \frac{1}{1+n^n} + \cdots.$

5. $1+1+\frac{5}{9}+\cdots+\frac{1+2^{n-1}}{3^{n-1}}+\cdots.$

6. $1+\frac{1}{2\cdot 2!}+\frac{1}{3\cdot 3!}+\cdots+\frac{1}{n\cdot n!}+\cdots.$

7. $\frac{1}{3!}+\frac{1}{5!}+\cdots+\frac{1}{(2n+1)!}+\cdots.$

8. $\frac{2}{3}+\frac{3}{4\cdot 2!}+\frac{4}{5\cdot 3!}+\cdots+\frac{n+1}{(n+2)\cdot n!}+\cdots.$

9. $1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}+\cdots+\frac{1}{n\cdot 2^{n-1}}.$

10. $1+\frac{1}{3}+\frac{1}{2\cdot 3^2}+\cdots+\frac{1}{(n-1)3^{n-1}}+\cdots.$

207. II. **Comparison test for divergence.** — *Let*

$$a_1+a_2+\cdots+a_n+\cdots$$

be an infinite series with positive terms. If a second infinite series with positive terms

$$b_1+b_2+\cdots+b_n+\cdots$$

is divergent, and if every term of this latter series is less than, or equal to, the corresponding term of the first series, then the first series is divergent.

Denote by s_n and S_n the sums of the first n terms of these two series respectively. Since S_n increases as n increases and does not approach a limit, it must increase beyond all limit (see fundamental principle). But for any value of n it is equal to, or less than, s_n. Hence s_n increases without limit and the first series is divergent.

In order to have a useful basis for comparison in the application of this test we shall prove that the series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$

is divergent.

Consider the n successive terms of this series,

$$\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}.$$

Every one of these terms is greater than $\frac{1}{2n}$ and their sum is therefore greater than $n \cdot \frac{1}{2n}=\frac{1}{2}$. Now the series contains an unlimited number of groups of terms like this with no terms in common. If then, we take n so great that we can form ten million groups of this kind, s_n will exceed five million; and by taking n sufficiently large we can make s_n exceed any number that we had in mind. Hence s_n does not approach any limit and the series is divergent.

This series is known as the *harmonic series* (see § 101).

We are now ready to illustrate the application of Test II.

Example. — Consider the series

$$1+\frac{3}{1\cdot 2}+\frac{5}{2\cdot 3}+\cdots+\frac{2n-1}{n(n-1)}+\cdots.$$

We observe that $$\frac{2n-1}{n(n-1)}=\frac{2n-1}{n-1}\cdot\frac{1}{n};$$

that is, that the nth term of this series is equal to $\frac{2n-1}{n-1}$ times the nth term of the harmonic series for all values of n greater than 1.

But when $n>1$, $\frac{2n-1}{n-1}>1$. Hence, by the comparison test, this series is divergent.

III. *If the nth term of a series does not approach zero as n increases without limit, the series is divergent.*

If a_n is the nth term of the series, then

$$a_n = s_n - s_{n-1};$$

and if the series is convergent, s_n and s_{n-1} approach the same limit as n increases without limit. Hence a_n approaches the limit zero.

If a_n approaches the limit zero as n increases without limit, it does not follow that the series is convergent. The harmonic series, for example, is divergent.

EXERCISES

Show that the following series are divergent:

1. $\frac{1}{6}+\frac{2}{11}+\cdots+\frac{n}{5n+1}+\cdots.$

2. $1+2!+3!+\cdots+n!+\cdots.$

3. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\cdots.$

4. $2+\frac{3}{2}+\frac{4}{3}+\cdots+\frac{n+1}{n}+\cdots.$

5. $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}+\cdots.$

6. $\frac{3}{2}+\frac{4}{5}+\cdots+\frac{2+n}{1+n^2}+\cdots.$

208. Since in the application of the next test we shall have occasion to look for the limit of certain fractions, we shall explain here, by means of examples, how to proceed in such cases.

Example 1. — Suppose that we want to know what happens to the fraction $\frac{2n+5}{3n+1}$ when n increases without limit.

We observe in the first place that both the numerator and the denominator increase without limit. But this gives us no information about the value of the fraction, since a fraction may have any value except 0, and have its numerator and denominator as large as we please. If, however, we first change the form of the fraction by dividing both the numerator and denominator by n, we shall be able to get the information we want.

$$\frac{2n+5}{3n+1}=\frac{2+\frac{5}{n}}{3+\frac{1}{n}}.$$

Now as n increases without limit the numerator of this last fraction approaches the limit 2 and the denominator approaches the limit 3, since $\frac{5}{n}$ and $\frac{1}{n}$ both approach zero. Hence the limit of the fraction is $\frac{2}{3}$.

Example 2.—Consider the limit of $\frac{n^2+3n+8}{2n-5}$ as n increases without limit.

$$\frac{n^2+3n+8}{2n-5}=\frac{1+\frac{3}{n}+\frac{8}{n^2}}{\frac{2}{n}-\frac{5}{n^2}}.$$

The numerator of this last fraction approaches the limit 1 and the denominator approaches the limit zero. Hence the fraction increases without limit.

Example 3.— Consider the limit of $\frac{3n-4}{8n^2-7n-2}$ as n increases without limit.

$$\frac{3n-4}{8n^2-7n-2}=\frac{\frac{3}{n}-\frac{4}{n^2}}{8-\frac{7}{n}-\frac{2}{n^2}}.$$

The numerator of this last fraction approaches the limit zero and the denominator approaches the limit 8. Hence the fraction approaches the limit zero.

EXERCISES

Find the limit of each of the following fractions as n increases without limit.

1. $\frac{n}{n+1}$.
2. $\frac{n}{n+2}$.
3. $\frac{n!}{(n+1)!}$.
4. $\frac{(n-1)3^{n-1}}{n\cdot 3^n}$.
5. $\frac{(2n-1)!}{(2n+1)!}$.
6. $\frac{3^{n^2}}{3^{(n+1)^2}}$.
7. $\frac{n\cdot n!}{(n+1)\cdot(n+1)!}$.
8. $\frac{2n-1}{2n+1}$.
9. $\frac{n^2+n+3}{n^2-n-3}$.
10. $\frac{7n+4}{n^2+1}$.

209. IV. **The ratio test.** — *If the terms of the infinite series*

$$a_1 + a_2 + \cdots + a_n + \cdots$$

are all positive, and if as n increases without limit the ratio $\frac{a_{n+1}}{a_n}$ approaches a limit l, the series is

(*a*) *convergent when* $l < 1$,

(*b*) *divergent when* $l > 1$.

If $l = 1$, *the series may be either convergent or divergent.*

(*a*) $l < 1$.

Select some number r that lies between l and 1 and is therefore less than 1. Since $\frac{a_{n+1}}{a_n}$ approaches the limit l as n increases without limit, we can select a positive interger m so large that $\frac{a_{n+1}}{a_n}$, for all values of n greater than m, differs from l by an amount less in absolute value than $r - l$. (See the definition of limit, § 202.)

For these values of n $\quad \frac{a_{n+1}}{a_n} < r$.

Hence, $\quad \frac{a_{m+2}}{a_{m+1}} < r$, or $a_{m+2} < r a_{m+1}$,

$$\frac{a_{m+3}}{a_{m+2}} < r, \text{ or } a_{m+3} < r a_{m+2} < r^2 a_{m+1},$$

. . . . . .

$$\frac{a_{m+p+1}}{a_{m+p}} < r, \text{ or } a_{m+p+1} < r a_{m+p} < r^p a_{m+1},$$

. . . . . .

Therefore every term of the series

$$a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_{m+p+1} + \cdots,$$

except the first one, is less than the corresponding term of the geometric progression

$$a_{m+1} + r a_{m+1} + r^2 a_{m+1} + \cdots + r^p a_{m+1} + \cdots,$$

which is convergent, since the common ratio r is less than 1 in absolute value. The series

$$a_{m+1} + a_{m+2} + \cdots + a_{m+p+1} + \cdots$$

is therefore convergent and hence the original series is convergent.

(*b*) $l > 1$.

As in (*a*), we can select a positive integer m so large that $\frac{a_{n+1}}{a_n}$, for all values of n greater than m, differs from l by an amount less in absolute value than $l-1$. For these values of n,

$$\frac{a_{n+1}}{a_n} > 1.$$

Hence, $\frac{a_{m+2}}{a_{m+1}} > 1$, or $a_{m+2} > a_{m+1}$,

$\frac{a_{m+3}}{a_{m+2}} > 1$, or $a_{m+3} > a_{m+2} > a_{m+1}$,

. . . . . .

$\frac{a_{m+p+1}}{a_{m+p}} > 1$, or $a_{m+p+1} > a_{m+p} > a_{m+1}$,

. . . . . .

Since these inequalities hold for all positive integral values of p, the series is divergent by Test III.

The harmonic series is an example of a divergent series for which $l = 1$; and the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$$

is an example of a convergent series for which $l = 1$. That this series is convergent may be seen by writing s_n in the form:

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Example. — Consider the series

$$1 + \tfrac{1}{3} + 2\,(\tfrac{1}{3})^2 + \cdots + (n-1)\,(\tfrac{1}{3})^{n-1} + \cdots.$$

$\frac{a_{n+1}}{a_n} = \frac{n\,(\frac{1}{3})^n}{(n-1)\,(\frac{1}{3})^{n-1}} = \frac{n}{3\,(n-1)}$, and the limit of this ratio as n increases without limit is $\frac{1}{3}$. Hence the series is convergent.

What has been said here about series all of whose terms are positive applies with slight, obvious modifications to series all of whose terms are negative.

In what way is it necessary to modify the *fundamental principle* in order that it shall apply to a variable that is always decreasing?

EXERCISES

Test the following series for convergence or divergence:

1. $3 + \dfrac{3^2}{2\,!} + \cdots + \dfrac{3^n}{n!} + \cdots.$

2. $\dfrac{1}{2\,!} + \dfrac{2}{3\,!} + \dfrac{3}{4\,!} + \cdots + \dfrac{n}{(n+1)!} + \cdots.$

3. $\dfrac{1}{10} + \dfrac{2\,!}{10^2} + \dfrac{3\,!}{10^3} + \cdots + \dfrac{n\,!}{10^n} + \cdots.$

4. $\dfrac{2 \cdot 3}{1} + \dfrac{3 \cdot 4}{2\,!} + \dfrac{4 \cdot 5}{3\,!} + \cdots + \dfrac{(n+1)(n+2)}{n\,!} + \cdots.$

5. $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} + \cdots.$

6. $1 + \dfrac{1}{2^2} + \dfrac{2\,!}{3^2} + \cdots + \dfrac{(n-1)\,!}{n^2} + \cdots$

7. $1 + 2 \cdot \tfrac{2}{3} + 3 \cdot (\tfrac{2}{3})^2 + \cdots + n\,(\tfrac{2}{3})^{n-1} + \cdots.$

8. $\dfrac{2}{1 \cdot 3} + \dfrac{2^3}{3 \cdot 3^3} + \dfrac{2^5}{5 \cdot 3^5} + \cdots + \dfrac{2^{2n-1}}{(2n-1) \cdot 3^{2n-1}} + \cdots.$

9. $2 + \dfrac{2^2}{2} + \dfrac{2^3}{3} + \cdots + \dfrac{2^n}{n} + \cdots.$

SERIES WITH POSITIVE AND NEGATIVE TERMS

210. Alternating series. — A series whose terms are alternately positive and negative is called an **alternating series.**

The following test applies to these series:

V. *An alternating series is convergent if the absolute value of every term is less than that of the preceding term and if the limit of the nth term is zero as n increases without limit.*

Thus the alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^{n+1}\frac{1}{n}+\cdots$$

is convergent.

We shall omit the proof of this test.

211. Absolutely convergent series. — The symbol $|a|$ represents the absolute value of a.

VI. *The series* $a_1+a_2+a_3+\cdots+a_n+\cdots$

is convergent if the series

$$|a_1|+|a_2|+|a_3|+\cdots+|a_n|+\cdots$$

is convergent.

We shall omit the proof of this statement.

If the series whose terms are the absolute values of the corresponding terms of a given series is convergent, the given series is said to be **absolutely convergent.**

The series $1-\frac{1}{2!}+\frac{1}{3!}-\cdots+(-1)^{n+1}\frac{1}{n!}+\cdots$

is absolutely convergent (see § 206).

Every convergent series whose terms are all of the same sign is absolutely convergent, but not all convergent series are absolutely convergent.

For example, the series

$$1-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^{n+1}\frac{1}{n}+\cdots.$$

212. VII. The general ratio test. — *If in the series*

$$a_1 + a_2 + \cdots + a_n + \cdots,$$

whose terms may be positive or negative, the absolute value of the ratio $\frac{a_{n+1}}{a_n}$ *approaches the limit* l *as* n *increases without limit, the series is*

(*a*) *convergent when* $l < 1$,

(*b*) *divergent when* $l > 1$.

If $l = 1$ *the series may be either convergent or divergent.*

(*a*) $l < 1$.

By Test IV the series is absolutely convergent and therefore, by Test VI, it is convergent.

(*b*) $l > 1$.

In this case a_n cannot approach 0 as n increases without limit, and therefore, by Test III, the series is divergent.

That the series may be either convergent or divergent when $l = 1$ is shown by the examples cited under Test IV.

The general ratio test includes Test IV as a special case. There are, of course, many other tests, which cannot be given here.

EXERCISES

Test the following series for convergence or divergence:

1. $\frac{2}{3} - \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} - \cdots + (-1)^{n+1}\frac{n+1}{n+2}\cdot\frac{1}{n} + \cdots.$

2. $.3 - \frac{.3^3}{3!} + \frac{.3^5}{5!} - \cdots + (-1)^{n+1}\frac{.3^{2n-1}}{(2n-1)!} + \cdots.$

3. $1 - \frac{.3^2}{2!} + \frac{.3^4}{4!} - \cdots + (-1)^{n+1}\frac{.3^{2n}}{(2n)!} + \cdots.$

4. $\left(1+\frac{1}{1}\right)^2 - \left(1+\frac{1}{2}\right)^2 + \left(1+\frac{1}{3}\right)^2 - \cdots + (-1)^{n-1}\left(1+\frac{1}{n}\right)^2 + \cdots.$

5. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots + (-1)^{n+1}\frac{1}{\sqrt{n}} + \cdots.$

6. $1 - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots + (-1)^{n+1}\frac{2^{2n-1}}{(2n-1)!} + \cdots.$

213. Power series. — A series of the form

$$c_1 + c_2x + c_3x^2 + \cdots + c_nx^{n-1} + \cdots,$$

where the c's are constants, is called a **power series.**

A power series may converge for all values of x, or it may diverge for all values of x except 0. These are the extreme cases. Usually such a series converges for some values of x and diverges for the other values.

Test VII is the most useful one for determining the values of x for which a power series converges.

Example. Consider the series

$$\frac{x}{3 \cdot 1} - \frac{x^3}{3^3 \cdot 3} + \frac{x^5}{3^5 \cdot 5} - \cdots + (-1)^{n+1}\frac{x^{2n-1}}{3^{2n-1}(2n-1)} + \cdots.$$

Here $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+1}}{3^{2n+1}(2n+1)} \cdot \frac{3^{2n-1}(2n-1)}{x^{2n-1}}\right| = \left|\frac{x^2(2n-1)}{3^2(2n+1)}\right|$.

The limit of this fraction as n increases without limit is $\frac{x^2}{3^2}$.

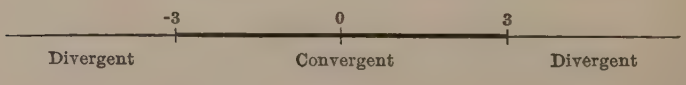
Hence the series is convergent for those values of x for which $\frac{x^2}{3^2} < 1$, or for which $-3 < x < 3$; and the series is divergent for those values of x for which $\frac{x^2}{3^2} > 1$. This inequality is satisfied whenever $x < -3$ or $x > 3$.

The test gives no information as to the convergence or divergence of the series when $x = -3$ or 3. These cases require a special investigation. It is easy to see directly that this particular series is convergent for either of these values of x.

The values of x for which the series is convergent form what is called the *interval of convergence* of the series.

Thus, the interval of convergence of the series just considered extends from -3 to 3.

We can indicate our conclusions graphically by making the interval of convergence heavier than the rest of the axis.



EXERCISES

Determine the interval of convergence of each of the following series, and represent it graphically:

1. $1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\cdots.$

2. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}+\cdots.$

3. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^{n+1}\frac{x^{2n}}{(2n)!}+\cdots.$

4. $x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^{n+1}\frac{x^{2n-1}}{2n-1}+\cdots.$

5. $1+x+x^2+\cdots+x^{n-1}+\cdots.$

6. $x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots.$

7. $x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\cdots.$

8. $\frac{9}{2}+\frac{10}{3}x+\frac{35}{12}x^2+\cdots+\frac{2n^2+5n+2}{n(n+1)}x^{n-1}+\cdots.$

APPENDIX

FORMULÆ FROM SOLID GEOMETRY

1. Section of a pyramid.—If a plane be drawn parallel to the base of a pyramid V–ABC, cutting the pyramid in the section DEF, and if S_1 and S_2 represent the areas of ABC and DEF respectively, then

$$\frac{S_1}{S_2} = \frac{\overline{VH}^2}{\overline{VK}^2},$$

where VH and VK are the distances of the base and the cutting plane respectively from the vertex. (For figure, see p. 37.)

2. Surface and volume of a sphere.—If S represents the surface, and V the volume, of a sphere of radius r, then

$$S = 4\pi r^2,$$

and

$$V = \tfrac{4}{3}\pi r^3.$$

3. Volume of a spherical segment of one base.—If a sphere be divided into two parts by a plane, either part is called a **spherical segment of one base.**

The section made by the plane is a circle and is called the **base** of the segment.

If we draw the radius of the sphere through the center of the base, the length of that part of the radius which lies between the base and the surface of the sphere is called the **altitude** of the segment.

If v represents the volume of the segment, x its altitude, and r_1 the radius of its base, then

$$v = \tfrac{1}{2}\pi x r_1^2 + \tfrac{1}{6}\pi x^3.$$

FORMULÆ FROM PHYSICS

1. Distance passed over by a falling body.—The distance s passed over by a falling body in t seconds is given in feet by the formula

$$s = 16\,t^2.$$

2. Velocity of a falling body.—The velocity v of a body that has fallen s feet is given in feet per second by the formula

$$v = \sqrt{64.3\,s}.$$

The formulæ in **1** and **2** are both approximate, but the latter is the more exact.

3. Velocity of a projectile.—The velocity v of a projectile at any moment in feet per second is given by the formula

$$v = \sqrt{v_0^2 - 64\,y},$$

where v_0 is the initial velocity and y is the height of the projectile at this moment in feet above the level of the starting point.

4. Number of vibrations made by a stretched wire.—The number n of vibrations made by a stretched wire is given by the formula

$$n = \frac{1}{2\,l}\sqrt{\frac{980\,M}{m}},$$

where l is the length in centimeters between the bridges, M the weight in grams of the stretching weight, and m the weight in grams of the wire per centimeter of length.

5. Oscillation of a pendulum.—The time t of oscillation of a pendulum of length l centimeters is given in seconds by the formula

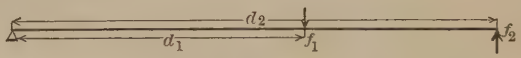
$$t = \pi\sqrt{\frac{l}{980}}.$$

6. The lever.—The point of support of a lever is called the **fulcrum.**

There are different kinds of levers, but in all of them two applied forces are in equilibrium (the effect of the weight of the lever being neglected) when the product of the first force and the distance of its point of application from the fulcrum is equal to the product of the second force and the distance of its point of application from the fulcrum.

$$f_1 d_1 = f_2 d_2.$$

The two forces must be applied in the same direction when their points of application are on opposite sides of the fulcrum, as in a teeter board. The forces must be applied in opposite directions when their points of application are on the same side of the fulcrum.



When two forces f_1 and f_2 act in the same direction on a lever at the distances d_1 and d_2 respectively from the fulcrum and on the same side of it, the combined effect is $f_1 d_1 + f_2 d_2$.

The effect of the weight of a uniform lever is the same as if the entire weight were concentrated at the middle point.

7. Center of gravity. — If the centers of gravity of two bodies of weights W_1 and W_2 respectively lie on the x-axis and have the abscissæ a and b respectively, then the abscissa c of the center of gravity of the two together is given by the formula

$$c = \frac{W_1 a + W_2 b}{W_1 + W_2}.$$

8. Specific gravity. — The ratio of the weight of a volume of a given substance to the weight of the same volume of water is called the **specific gravity** of the substance.

Thus, a cubic inch of platinum weighs approximately 22 times as much as a cubic inch of water, and we accordingly say that the specific gravity of platinum is approximately 22.

9. Principle of Archimedes. — A body immersed in a liquid loses a part of its weight equal to the weight of the displaced liquid.

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J. C. M.

JCM.

$$mx = nw - nx$$

$$mx + nx = nw$$

$$x = \frac{nw}{m-n}$$

Love is just like the rheumatism, unless you get it out of your
Lblood it is liable to break out in a fresh place.

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G. W. Cox
B. R. Carver
W. D. Haynes
J. R. White

S. E. Bratton
W. M. Thompson
A. T. Love
B. L. Stookspury
R. L. Queen
A. T. McNay
R. E. Jefferies
P. R. Pilkenton
H. G. Vawter
C. L. Bond
W. Bond
C. T. Miller
E. L. Ball
A. R. Pilkenton
O. P. Francisco
R. G. Farmer
L. E. Hughes
V. H. Francisco
L. W. McFarland